Dynamic Information Fusion with the Integration of Local Observers, Value of Information, and Active-Passive Consensus Filters

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This paper proposes a dynamic information fusion framework for sensor networks with the integration of local observers, value of information, and active-passive consensus filters as well as a layer to monitor the validity of information. Specifically, we consider a process of interest consisting of multiple subprocesses (for example, multiple targets to be monitored). The heterogeneity in the sensor networks is considered and handled in many aspects such as nodes are allowed to have different sensing capabilities, different information node roles (active and/or passive; that is, a node can be subject to observations of the process or to no observation), and different weights on information (value of information). In addition, the information validity monitor layer allows operators to evaluate the reliability of the fused information based on the local feedbacks received from the sensor network. Several illustrative numerical examples are also presented to illustrate the efficacy and discuss the practical aspects of the proposed dynamic information fusion framework.

I. Introduction

With the remarkable technological developments in the past two decades, advanced devices such as autonomous mobile robots, and sensors have become affordable to deploy in a large quantities. This also leads to a need in the development of advanced algorithms to gather and integrate information as well as to control such multiagent systems. In particular, dynamic information fusion in sensor networks plays an important roles in a wide array of applications for both scientific, civilian and military purposes. One of the main challenges in dynamic information fusion is the heterogeneity of sensor networks. The sources of this heterogeneity include the differences in sensor modalities, the quality of sensing information (value of information), and the information roles of nodes (active and passive; that is, a node can be subject to observations of the process or to no observation), to name but a few examples.

While information roles of nodes and active-passive consensus filter are recently investigated in Refs. 1–5 and references therein, these results often consider scalar integrator dynamics and/or lack a complete

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structure to process the local information before the fusion such as utilizing local observers to extract more information and assigning weights on information. Although the distributed algorithms in Refs. 6 and 7 consider the differences in sensor modalities, the information roles of nodes in these results are not explicitly discussed. Several works such as Refs. 8–10 consider the value of information, yet information roles of nodes and/or heterogeneous modalities are not considered. Nonetheless, the aforementioned works do not have a direct architecture to quantify and evaluate the quality of fused information of sensor networks in real-time.

The contribution of this paper is to propose a dynamic information fusion framework for sensor networks with the integration of local observers, value of information, and active-passive consensus filters as well as a layer to monitor the validity of information; see Figure 1. Specifically, we consider a process of interest consists of multiple subprocesses (e.g, multiple targets to be monitored). The heterogeneity in the sensor networks is considered and handled in many aspects such as nodes are allowed to have different sensing capabilities, different information node roles (active and passive), and different weights on information (value of information). In addition, the information validity monitor layer allows operators to evaluate the reliability of the fused information based on the local feedbacks received from the sensor network. Several illustrative numerical examples are also presented to illustrate the efficacy and discuss the practical aspects of the proposed dynamic information fusion framework.

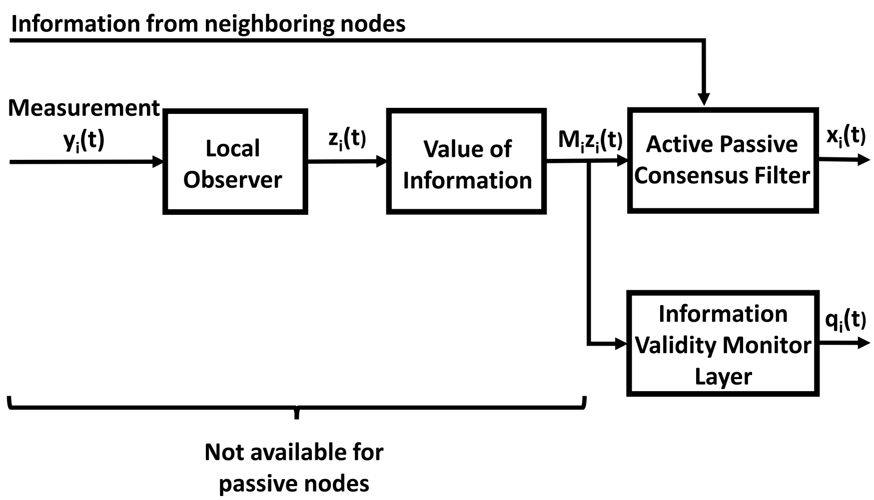

Figure 1. The dynamic information fusion framework of an individual node with the integration of a local observer, value of information, active-passive consensus filter, and information validity monitor layer.

The organization of this paper is as follows. In Section II, we present the setup of the process, the sensor network, the structure of the local observers, and the value of information. In Section III, the active-passive consensus filter with fixed information node roles is introduced and analyzed, as an intermediate result. The main result of this paper is then presented in Section IV, which is a practical extension of the result of Section III to the time-varying case. The information validity monitor layer is presented in Section V and illustrative numerical examples together with some discussions are provided in Section VI. Finally, concluding remarks are summarized in Section VII. For readers, we refer to Appendix A for the notations, mathematical preliminaries, and necessary lemmas for the results of this paper.

II. Problem Setup

A. Considered Process and the Sensor Network

Consider a large-scale process of interest with the dynamics given by

$$\dot{z} = Az(t), \; z(0) = z_0, \tag{1}$$

where $z(t) \in \mathbb{R}^n$ denotes the unmeasurable process state vector,

$$A = \text{block–diag}(A_1, A_2, \ldots, A_m) \in \mathbb{R}^{n \times n}, \tag{2}$$

is the system matrix with $A_h \in \mathbb{R}^{n_h \times n_h}$ for $h = 1, 2, \ldots, m$ and $n_h \leq n, \sum_{h=1}^{m} n_h = n$. While (1) adopts a simple structure in order to allow us to directly focus on the overarching contribution of this paper (i.e., a new dynamic information fusion framework), it can represent linear or linearized, controlled or uncontrolled process dynamics. Note that the contribution of this paper can be readily extended to the cases, where (1) include measurement noise and/or uncertainties resulting from modeling efforts. Furthermore, the block-diagonal structure of the system matrix A indicates that the process can be decomposed into m subprocesses. An example of such a process is independent multiple targets that need to be monitored on an observation field, which presents the overarching application focus of this paper.

Next, consider a sensor network with N nodes exchanging information among each other using their local measurements through a connected, undirected graph $\mathcal{G}$. If a node i, $i = 1, \ldots, N$ has no observations, then we say that it is a "passive node". On the other hand, if a node i, $i = 1, \ldots, N$ is subject to observations of process (1) given by

$$y_i(t) = {}^{g}C_i z(t), \tag{3}$$

where $y_i(t) \in \mathbb{R}^p$ denotes the measurable process output for node i, $i = 1, \ldots, N$, and ${}^{g}C_i \in \mathbb{R}^{p \times n}$ denotes the system output matrix with g is the sensor's category defined below, then we say that node i is an "active node".

Remark 1. *Here, we consider that the sensors can have different sensing capabilities and can be categorized into multiple categories. We define a Category I sensor as* ${}^{I}C_i = \begin{bmatrix} 0 & \ldots & {}^{j}\bar{C}_i & \ldots & 0 \end{bmatrix}$ *such that* ${}^{j}\bar{C}_i \in \mathbb{R}^{p \times n_j}$ *and the pair* $(A_j, {}^{j}\bar{C}_i)$ *is detectable, where* $j \in \mathbb{Z}_+$ *denotes the corresponding subprocess that node* i *can sense, and* $j \in [1, m]$ *(e.g., a Category I sensor can observe a specific subprocess). Next, we define a Category II sensor as a combination of two or more Category I sensors, for example,* ${}^{II}C_i = \begin{bmatrix} 0 & \ldots & {}^{j}\bar{C}_i & \ldots & {}^{k}\bar{C}_i & \ldots & 0 \end{bmatrix}$ *such that* ${}^{j}\bar{C}_i \in \mathbb{R}^{p \times n_j}$, ${}^{k}\bar{C}_i \in \mathbb{R}^{p \times n_k}$ *and the pairs* $(A_j, {}^{j}\bar{C}_i)$, $(A_k, {}^{k}\bar{C}_i)$ *are detectable, where* $j, k \in \mathbb{Z}_+$ *and* $j, k \in [1, m]$ *(e.g., a Category II sensor can simultaneously observe several subprocesses). We also define a Category III sensor as a generalized sensor that does not possess a detectable pair (or an observable pair), for example,* ${}^{III}C_i = \begin{bmatrix} 0 & \ldots & {}^{j}\bar{c}_i & \ldots & {}^{k}\bar{c}_i & \ldots & 0 \end{bmatrix}$*, where* ${}^{j}\bar{c}_i, {}^{k}\bar{c}_i \in \mathbb{R}$, $j, k \in \mathbb{Z}_+$ *and* $j, k \in [1, n]$ *(e.g, a Category III sensor can observe a state of a subprocess or several states of several subprocesses). A sensor's category is not necessarily limited to the ones that we* *introduce above but can be extended to the mixtures of those categories as well.*

B. Local Observers and the Value of Information Matrix

We first introduce here the construction of local observers based on the local measurements $y_i(t)$. Depending on the sensor's category, the value of information matrix is then constructed. Because of the diagonal structure of the system matrix A, we can construct the local observer vector $z_i(t) \in \mathbb{R}^n$ for the process based on the type and capability of each active node i. Specifically, if the sensor is in either Category I or II, a subprocess state can be estimated by the local (Luenberger) observer given by

$$\dot{s}_i(t) = A_h s_i(t) + {}^h L_i(y_i(t) - {}^h\bar{C}_i s_i(t)),\ s_i(0) = s_{i0}, \tag{4}$$

where h is an index of a corresponding subprocess of A that node i can observe, $s_i \in \mathbb{R}^{n_h}$ is the local state estimate of a subprocess A_h, ${}^h\bar{C}_i$ is the output matrix corresponding to states observed by node i on the subprocess A_h, and ${}^h L_i \in \mathbb{R}^{n_h \times p}$ is the corresponding local observer gain for node i. From this point, the local observer of node i, $z_i(t) \in \mathbb{R}^n$ can be constructed, for example $z_i(t) \triangleq \begin{bmatrix} 0 & \ldots & s_i^{\mathrm{T}}(t) & \ldots & 0 \end{bmatrix}^{\mathrm{T}}$, where the position of $s_i(t)$ is corresponding to the state of the subprocess A_h in $z(t)$. If the sensor is in Category III, there is no theoretical need for a local observer. Therefore, $z_i(t)$ can be constructed directly from $y_i(t)$ as $z_i(t) \triangleq \begin{bmatrix} 0 & \ldots & y_i(t)^{\mathrm{T}} & \ldots & 0 \end{bmatrix}^{\mathrm{T}}$, where the position of $y_i(t)$ elements are corresponding to the substate of $z(t)$ that node i can sense.

Based on the sensor's type and capability, in addition, the value of information matrix has a natural diagonal structure in the form given by

$$M_i \triangleq \mathrm{diag}(m_i) \in \mathbb{R}^{n \times n}, \tag{5}$$

where $m_i \triangleq \begin{bmatrix} m_{i1} & m_{i2} & \ldots & m_{in} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ for $i = 1, 2, \ldots, N$ and m_{ir} are nonnegative scalar weights with $r = 1, 2, \ldots, n$. A substate of $z_i(t)$ is called "valid" when its weight is positive. Conversely, a substate of $z_i(t)$ is called "invalid" when its weight is 0. Under certain circumstances (e.g., see Ref. 8 and references therein), a sensor can be subject to some observations, yet the information may not be reliable (and so its substates are set to be invalid). We also assume that $z_i(t)$ and $\dot{z}_i(t)$ are bounded. Considering the multivehicle application focus of this paper as the process to be monitored, this assumption generally holds, because the vehicles' properties such as positions and velocities are bounded, on the observation field. After $z_i(t)$ and the value of information matrix M_i are constructed, they are then passed to the active-passive consensus filter for information fusion and to the information validity monitor layer for evaluation of the quality of fused information as illustrated in Figure 1.

For the purpose of establishing an intermediate result, Section III next presents the active-passive consensus filter for the case where the active-passive roles of nodes are fixed for each node and assume the local estimation $z_i(t)$ is constant. We then introduce the main result of this paper by extending the result of Section III to the actual practical case, where both the active-passive role of each node and $z_i(t)$ are time-varying in Section IV.

III. Active-Passive Consensus Filters with Fixed Information Node Roles

A. Proposed Architecture

The active-passive consensus filter aims to drive substates of each node to the average of all valid active corresponding substates (i.e, an agent needs to be active and its corresponding substates need to have positive weights) of the vectors of the local observers $z_i(t)$, $i = 1, 2, \ldots, N$. Specifically, in this section, we assume that the active-passive role of each node is fixed and the vectors of the local observers $z_i(t) \equiv z_i$, $i = 1, 2, \ldots, N$, are constants for the sake of establishing an intermediate result for the following sections of this paper. Mathematically speaking, we consider the proposed active-passive consensus filter given by

$$\begin{aligned} \dot{x}_i(t) &= -\alpha \sum_{i \sim j} (x_i(t) - x_j(t)) + \alpha \sum_{i \sim j} (\xi_i(t) - \xi_j(t)) - \alpha k_i M_i (x_i(t) - z_i), \ x_i(0) = x_{i0}, \quad (6) \\ \dot{\xi}_i(t) &= -\gamma \sum_{i \sim j} (x_i(t) - x_j(t)), \ \xi_i(0) = \xi_{i0}, \quad (7) \end{aligned}$$

where $x_i(t) \in \mathbb{R}^n$, $\xi_i(t) \in \mathbb{R}^n$, $z_i \in \mathbb{R}^n$ denote the state, the integral action, and the local observer vector of node i, $i = 1, \ldots, N$, respectively. Here, M_i is the value of information matrix defined in (5) and $\alpha, \gamma \in \mathbb{R}_+$ are constant gains. Under the assumption that the information node roles are fixed, $k_i = 1$ for active nodes and $k_i = 0$ for passive nodes.

Remark 2. *We introduce a similar active-passive consensus filter in Ref. 8; however, our previous result documented in that paper only considered scalar integrator dynamics. We next present the stability properties of (6) and (7) having $x_i(t) \in \mathbb{R}^n$ and $\xi_i(t) \in \mathbb{R}^n$. In addition, we note that the stability results documented in Ref. 8 can also be applied to each scalar element of (6) and (7) in parallel. Yet, the presented stability properties of the next subsection is compact in the sense that we do not focus on scalar elements of (6) and (7) but to their compact form; hence, from our standpoint, it is worth to include the following content to this paper for completeness.*

B. Stability Analysis

Let $x(t) \triangleq \begin{bmatrix} x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t) & \ldots & x_N^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, $\xi(t) \triangleq \begin{bmatrix} \xi_1^{\mathrm{T}}(t) & \xi_2^{\mathrm{T}}(t) & \ldots & \xi_N^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, and $\zeta \triangleq \begin{bmatrix} z_1^{\mathrm{T}} & z_2^{\mathrm{T}} & \ldots & z_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The proposed algorithm (6) and (7) can be rewritten in the compact form given by

$$\begin{aligned} \dot{x}(t) &= -\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) x(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) \xi(t) - \alpha \mathcal{M} x(t) + \alpha \mathcal{M} \zeta \\ &= -\alpha F x(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) \xi(t) + \alpha \mathcal{M} \zeta, \quad x(0) = x_0, \quad (8) \\ \dot{\xi}(t) &= -\gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) x(t), \quad \xi(0) = \xi_0, \quad (9) \end{aligned}$$

where

$$\mathcal{M} \triangleq \text{block-diag}(k_1 M_1, k_2 M_2, \ldots, k_N M_N) \in \mathbb{R}^{Nn \times Nn}, \quad (10)$$

and $F \triangleq \big(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n + \mathcal{M}\big) \in \mathbb{R}^{Nn \times Nn}$. Since M_i for all $i = 1, 2, \ldots, N$ are diagonal matrices, $\mathcal{M}$ is then a diagonal matrix. Furthermore, since $(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)$ and $\mathcal{M}$ are nonnegative definite, F is either nonnegative definite or positive definite.

Note that a substate of z_i is said to be valid if its weight in M_i is positive. In addition, z_i is active when $k_i = 1$. Since we are interested in driving substates of $x_i(t)$ of each node to the average of all valid active corresponding substates of local observer vectors z_i , $i = 1, 2, \ldots, N$ in the network, define

$$\begin{aligned} S &\triangleq (\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}(\mathbf{1}_N \otimes \mathrm{I}_n) \in \mathbb{R}^{n \times n} \\ &= k_1 M_1 + k_2 M_2 + \ldots + k_N M_N, \end{aligned} \tag{11}$$

as the diagonal matrix with the total weight of valid active substates of z_i on the diagonal. We now let

$$\epsilon \triangleq S^{+}(\mathbf{1}_N^{T} \otimes \mathrm{I}_n)\mathcal{M}\zeta \in \mathbb{R}^n \tag{12}$$

be the average of all valid active substates of local observer vectors z_i, $i = 1, 2, \ldots, N$ in the network. Note that since z_i, $i = 1, 2, \ldots, N$ are constant, ζ and ϵ are also constant in this case.

Remark 3. *In this section, we do not assume collective observability (see, for example, Refs. 7, 11). Collective observability can imply that at any moment there exists at least one node that is active and has valid information for each substate of z_i; hence, the matrix S has full rank and is invertible. Recall that S is a diagonal matrix with each element on the diagonal is the total weight of the corresponding valid active substate of local observer vectors z_i. In addition, a substate of $x_i(t)$ and $z_i(t)$ are said to be completely passive if there is no node in the network can observe or has a valid observation on that corresponding substate. In other words, if the substate r is completely passive (that is, $k_i m_{ir} = 0$ for all nodes $i = 1, \ldots, N$ where m_{ir} is the r-th element on the diagonal of the matrix M_i), then S has a row of zeros (the r-th row). A nice property of the pseudo-inverse of a diagonal matrix is that each positive diagonal element is inversed, except zero diagonal elements are still zeros. For example, if $S = \mathrm{diag}\left(\begin{bmatrix} a & b & 0 \end{bmatrix}\right)$, then $S^{+} = \mathrm{diag}\left(\begin{bmatrix} a^{-1} & b^{-1} & 0 \end{bmatrix}\right)$ for $a, b \in \mathbb{R}_{+}$. In the case if a substate of z_i is completely passive, the corresponding substate of $x_i(t)$ converges to average consensus and this happens only when an element on the diagonal of S is 0. This is shown and discussed later in Remark 5 of Section III.C.*

We now define the error $\delta(t) \triangleq \big(x(t) - (\mathbf{1}_N \otimes \epsilon)\big) \in \mathbb{R}^{Nn}$ and taking its time derivative to obtain

$$\begin{aligned} \dot{\delta}(t) &= -\alpha F\big(\delta(t) + (\mathbf{1}_N \otimes \epsilon)\big) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\xi(t) + \alpha\mathcal{M}\zeta \\ &= -\alpha F\delta(t) - \alpha\mathcal{M}(\mathbf{1}_N \otimes \epsilon) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\xi(t) + \alpha\mathcal{M}\zeta \\ &= -\alpha F\delta(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\xi(t) - \alpha\mathcal{M}\omega, \ \delta(0) = \delta_0, \end{aligned} \tag{13}$$

where the third equality comes from the facts that $F = \big(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n + \mathcal{M}\big)$ and $\mathcal{L}(\mathcal{G})\mathbf{1}_N = \mathbf{0}_N$, and $\omega \triangleq \big((\mathbf{1}_N \otimes \epsilon) - \zeta\big) \in \mathbb{R}^{Nn}$. Next, define $e(t) \triangleq \big(\xi(t) - \frac{\alpha}{\gamma}(\mathcal{L}^{+}(\mathcal{G}) \otimes \mathrm{I}_n)\mathcal{M}\omega\big) \in \mathbb{R}^{Nn}$ and take its time derivative as

$$\dot{e}(t) = -\gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\big(\delta(t) + (\mathbf{1}_N \otimes \epsilon)\big)$$

$$= \quad -\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)\delta(t),\ e(0)=e_0. \tag{14}$$

From (13), Lemma A.1, and the definition of $e(t)$, one can write

$$\begin{aligned}
\dot{\delta}(t) &= -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)\Big(e(t)+\frac{\alpha}{\gamma}(\mathcal{L}^{+}(\mathcal{G})\otimes \mathrm{I}_n)\mathcal{M}\omega\Big)-\alpha\mathcal{M}\omega \\
&= -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t)+\alpha\Big(\big(\mathcal{L}(\mathcal{G})\mathcal{L}^{+}(\mathcal{G})\otimes \mathrm{I}_n\big)-\mathrm{I}_{Nn}\Big)\mathcal{M}\omega \\
&= -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t)+\alpha\Big(\big((\mathrm{I}_N-\tfrac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}})\otimes \mathrm{I}_n\big)-\mathrm{I}_{Nn}\Big)\mathcal{M}\omega \\
&= -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t)+\alpha\Big(\big(\mathrm{I}_{Nn}-(\tfrac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}})\otimes \mathrm{I}_n\big)-\mathrm{I}_{Nn}\Big)\mathcal{M}\omega \\
&= -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t)-\frac{\alpha}{N}(\mathbf{1}_N\otimes \mathrm{I}_n)(\mathbf{1}_N^{\mathrm{T}}\otimes \mathrm{I}_n)\mathcal{M}\omega,\ \delta(0)=\delta_0,
\end{aligned} \tag{15}$$

To further write (15) in a simpler form, we now introduce the following lemma.

Lemma 1. *Let ϵ be defined by (12), $\mathcal{M}$ be defined by (10), and $\omega \triangleq \big((\mathbf{1}_N\otimes\epsilon)-\zeta\big)$. Then,*

$$(\mathbf{1}_N^{\mathrm{T}}\otimes \mathrm{I}_n)\mathcal{M}\omega = 0. \tag{16}$$

Proof. See the Appendix B.

Remark 4. *Although, owing to the assumptions of this subsection, k_i, ζ, and ϵ are constant, the result of Lemma 1 is still valid for the case when $k_i(t)$, $\zeta(t)$, and $\epsilon(t)$ are time-varying.*

Under the result of Lemma 1, (15) can now be simplified as

$$\dot{\delta}(t) = -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t),\ \delta(0)=\delta_0. \tag{17}$$

The closed-loop error dynamics of the system given by (6) and (7) are

$$\dot{\delta}(t) = -\alpha F\delta(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t),\ \delta(0)=\delta_0. \tag{18}$$

$$\dot{e}(t) = -\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)\delta(t),\ e(0)=e_0. \tag{19}$$

We are now ready to state the following result.

Theorem 1. *Consider a sensor network with N nodes given by (6) and (7), where nodes exchange information using local measurements under a connected, undirected graph $\mathcal{G}$. Then, the closed loop error dynamics (18) and (19) are Lyapunov stable and $\delta(t)$ converges to the null space of F.*

Proof. Consider the Lyapunov function candidate given by

$$V(\delta,e) = \frac{1}{2}\delta^{\mathrm{T}}\delta+\frac{1}{2}e^{\mathrm{T}}e. \tag{20}$$

Note that $V(0,0)=0$ and $V(\delta,e)>0$ for all $(\delta,e)\neq 0$. The time derivative of (20) along the trajectories of

(18) and (19) is given by

$$\begin{aligned}\dot{V}(\delta(t), e(t)) &= -\alpha\delta^{\mathrm{T}}(t)F\delta(t) + \gamma\delta^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)e(t) - \gamma e^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)\delta(t) \\ &= -\alpha\delta^{\mathrm{T}}(t)F\delta(t) \leq 0 \end{aligned} \tag{21}$$

Therefore, the closed-loop error dynamics (18) and (19) are Lyapunov stable. Because $\ddot{V}(\delta(t), e(t))$ is also bounded for all $t \geq 0$, it follows from Barbalat's lemma (see Ref. 12) that $\lim_{t\to\infty} \dot{V}(\delta(t), e(t)) = \lim_{t\to\infty} \Big(-\alpha\delta^{\mathrm{T}}(t)F\delta(t)\Big) = 0$. Therefore, as $t \to \infty$,

$$\delta^{\mathrm{T}}(t)F\delta(t) = \delta^{\mathrm{T}}(t)F^{1/2}F^{1/2}\delta(t) = \big(F^{1/2}\delta(t)\big)^{\mathrm{T}}\big(F^{1/2}\delta(t)\big) = \|\big(F^{1/2}\delta(t)\big)\|_2 \to 0. \tag{22}$$

Note that (22) indicates that $\delta(t)$ converges to the null space of $F^{1/2}$. Since F is a symmetric matrix, it is always diagonalizable by an orthogonal matrix $U \in \mathbb{R}^{Nn\times Nn}$ such that $F = U\Lambda U^{\mathrm{T}}$, where $\Lambda \in \mathbb{R}^{Nn\times Nn}$ is the diagonal matrix with eigenvalues of F on the diagonal. As a result, $F^{1/2} = U\Lambda^{1/2}U^{\mathrm{T}}$ since $F^{1/2}F^{1/2} = U\Lambda^{1/2}U^{\mathrm{T}}U\Lambda^{1/2}U^{\mathrm{T}} = U\Lambda U^{\mathrm{T}} = F$. Therefore, $(F^{1/2})^{\mathrm{T}} = (U\Lambda^{1/2}U^{\mathrm{T}})^{\mathrm{T}} = (U^{\mathrm{T}})^{\mathrm{T}}\Lambda^{1/2}U^{\mathrm{T}} = U\Lambda^{1/2}U^{\mathrm{T}} = F^{1/2}$, thus $F^{1/2}$ is also a symmetric matrix. Utilize the result of Lemma A.4 for $F^{1/2}$, we have $\mathcal{N}(F^{1/2}) = \mathcal{N}(F)$. Hence, $\delta(t)$ converges to the null space of F. ■

We now investigate the null space of $F = \big(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n + \mathcal{M}\big)$ in the next subsection owing to the fact that the above theorem shows that $\delta(t)$ converges to the null space of F.

C. Convergence Analysis

We first decompose the structure of F as

$$\begin{aligned} F &= \mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n + \mathcal{M} \\ &= \begin{bmatrix} \mathcal{L}_{11}\mathrm{I}_n & \mathcal{L}_{12}\mathrm{I}_n & \dots & \mathcal{L}_{1N}\mathrm{I}_n \\ \mathcal{L}_{21}\mathrm{I}_n & \mathcal{L}_{22}\mathrm{I}_n & \dots & \mathcal{L}_{2N}\mathrm{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{N1}\mathrm{I}_n & \mathcal{L}_{N2}\mathrm{I}_n & \dots & \mathcal{L}_{NN}\mathrm{I}_n \end{bmatrix} + \begin{bmatrix} k_1M_1 & 0 & \cdots & 0 \\ 0 & k_2M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_NM_N \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11}\mathrm{I}_n + k_1M_1 & \mathcal{L}_{12}\mathrm{I}_n & \dots & \mathcal{L}_{1N}\mathrm{I}_n \\ \mathcal{L}_{21}\mathrm{I}_n & \mathcal{L}_{22}\mathrm{I}_n + k_2M_2 & \dots & \mathcal{L}_{2N}\mathrm{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{N1}\mathrm{I}_n & \mathcal{L}_{N2}\mathrm{I}_n & \dots & \mathcal{L}_{NN}\mathrm{I}_n + k_NM_N \end{bmatrix}, \end{aligned} \tag{23}$$

where $\mathcal{L}_{ij}$ denotes the corresponding element at i-th row and j-th column of the Laplacian matrix $\mathcal{L}(\mathcal{G})$. Recall that $M_i = \mathrm{diag}(m_i)$ is the value of information matrix of node i for $i = 1, 2, \dots, N$, where $m_i \triangleq \begin{bmatrix} m_{i1} & m_{i2} & \dots & m_{in}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ and m_{ir} are nonnegative scalar weights with $r = 1, 2, \dots, n$. Therefore, $\mathcal{M}$

can be rewritten as

$$\mathcal{M} = \mathrm{diag}\left(\begin{bmatrix} k_1 m_{11} & \dots & k_1 m_{1n} & \dots & k_n m_{N1} & \dots & k_n m_{Nn} \end{bmatrix}^{\mathrm{T}}\right). \tag{24}$$

Next, there is a permutation matrix $J \in \mathbb{R}^{Nn\times Nn}$ that allows us to reorder $\delta(t)$ as

$$J\delta(t) = J\begin{bmatrix} \delta_{11}(t) \\ \vdots \\ \delta_{1n}(t) \\ \vdots \\ \delta_{N1}(t) \\ \vdots \\ \delta_{Nn}(t) \end{bmatrix} = \begin{bmatrix} \delta_{11}(t) \\ \vdots \\ \delta_{N1}(t) \\ \vdots \\ \delta_{1n}(t) \\ \vdots \\ \delta_{Nn}(t) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{Substate 1} \\ \vdots \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{Substate n} \end{matrix} \tag{25}$$

where $\delta_{ir}(t)$ indicates the error of substate r, $r = 1, 2, \dots, n$ of node i, $i = 1, 2, \dots, N$. Therefore,

$$\begin{aligned} JFJ^{\mathrm{T}} &= J(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)J^{\mathrm{T}} + J\mathcal{M}J^{\mathrm{T}} \\ &= \begin{bmatrix} \mathcal{L}(\mathcal{G}) & & \\ & \ddots & \\ & & \mathcal{L}(\mathcal{G}) \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} k_1 m_{11} & & \\ & \ddots & \\ & & k_N m_{N1} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} k_1 m_{1n} & & \\ & \ddots & \\ & & k_N m_{Nn} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \mathcal{L}_{11} + k_1 m_{11} & \cdots & \mathcal{L}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{N1} & \cdots & \mathcal{L}_{NN} + k_N m_{N1} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} \mathcal{L}_{11} + k_1 m_{1n} & \cdots & \mathcal{L}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{N1} & \cdots & \mathcal{L}_{NN} + k_N m_{Nn} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_n \end{bmatrix}, \end{aligned} \tag{26}$$

where $F_r \triangleq \begin{bmatrix} \mathcal{L}_{11} + k_1 m_{1r} & \cdots & \mathcal{L}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{N1} & \cdots & \mathcal{L}_{NN} + k_N m_{Nr} \end{bmatrix}$ for $r = 1, 2, \dots, n$. Note that, from, for example, Lemma 2 in Ref. 13 or Lemma 3.3 in Ref. 14, if there exists at least one node i for $i = 1, 2, \dots, N$ such that $k_i m_{ir}$ is positive, F_r is positive definite and $\mathcal{N}(F_r) = 0$. On the other hand, if a substate r is completely passive, then $k_i m_{ir} = 0$ for all i as discussed in Remark 3. In this case, $F_r = \mathcal{L}(\mathcal{G})$, and hence, $\mathcal{N}(F_r) = \mathcal{N}(\mathcal{L}(\mathcal{G})) = \text{span}(\mathbf{1}_N)$.

Let $f(r)$ for $r = 1, 2, \ldots, n$ be a function such that

$$f(r) \triangleq \begin{cases} 0 & \text{if there exists at least one node } i \text{ in the network} \\ & \text{such that } k_i m_{ir} \text{ is positive,} \\ 1 & \text{otherwise.} \end{cases} \tag{27}$$

We now can write

$$\mathcal{N}(F_r) = \text{span}\big(f(r)\mathbf{1}_N\big). \tag{28}$$

Therefore,

$$\mathcal{N}(JFJ^{\mathrm{T}}) = \text{span}\left(\begin{bmatrix} f(1)\mathbf{1}_N \\ \vdots \\ f(n)\mathbf{1}_N \end{bmatrix}\right) = \text{span}(\nu) = a\nu, \tag{29}$$

where $a \in \mathbb{R}$. Note that if each specific substate r has at least one positive scalar weight m_{ir} for for some $i \in \mathcal{V}_{\mathcal{G}}$, then ν is a zero vector. In all other cases, $\nu \in \mathbb{R}^{Nn}$ is a non-zero vector.

Since J is invertible, $\text{rank}(JFJ^{\mathrm{T}}) = \text{rank}(JF) = \text{rank}(F)$. In addition, $\text{rank}(JFJ^{\mathrm{T}}) + \text{def}(JFJ^{\mathrm{T}}) = \text{rank}(JF) + \text{def}(JF) = \text{rank}(F) + \text{def}(F) = Nn$. Therefore, $\text{def}(JFJ^{\mathrm{T}}) = \text{def}(JF) = \text{def}(F)$. As a result, $(JF)J^{\mathrm{T}}(a\nu) = 0$ also indicates that $J^{\mathrm{T}}(a\nu)$ is the null space of JF. It should be also noted that the permutation matrix J satisfies $JJ^{\mathrm{T}} = J^{\mathrm{T}}J = \mathrm{I}_{Nn}$; hence, $J^{\mathrm{T}}JFJ^{\mathrm{T}}(a\nu) = (J^{\mathrm{T}}J)F(J^{\mathrm{T}}(a\nu)) = F(J^{\mathrm{T}}(a\nu)) = 0$. Consequently, $J^{\mathrm{T}}(a\nu)$ is the null space of F and we can rewrite $J^{\mathrm{T}}(a\nu)$ as

$$\eta \triangleq J^{\mathrm{T}}(a\nu) = \begin{bmatrix} f(1) \\ \vdots \\ f(n) \\ \vdots \\ f(1) \\ \vdots \\ f(n) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} f(1) \\ \vdots \\ f(n) \end{matrix}}\right\}\text{Node 1} \\ \vdots \\ \left.\vphantom{\begin{matrix} f(1) \\ \vdots \\ f(n) \end{matrix}}\right\}\text{Node N} \end{matrix} = a\left(\mathbf{1}_N \otimes \begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix}\right) \triangleq a(\mathbf{1}_N \otimes \bar{f}). \tag{30}$$

Note that since F is a constant matrix, $\bar{f} \in \mathbb{R}^n$ is also a constant vector in this case. In addition, Theorem 1 indicates that $\delta(t)$ converges to $a(\mathbf{1}_N \otimes \bar{f})$. Recall that by definition $\delta(t) \triangleq x(t) - (\mathbf{1}_N \otimes \epsilon)$, and thus

$$\lim_{t\to\infty} \big(x(t) - (\mathbf{1}_N \otimes \epsilon) - a(\mathbf{1}_N \otimes \bar{f})\big) \quad = \quad 0, \tag{31}$$

or equivalently,

$$\lim_{t\to\infty}\big(x(t)-(\mathbf{1}_N\otimes(\epsilon+a\bar{f})\big) \quad = \quad 0. \tag{32}$$

In general, (32) shows that under the proposed active-passive consensus filter given by (6) and (7) in subsection A, all nodes reach the consensus in substate-wise.

Remark 5. *If a substate r, $r=1,2,\ldots,n$ is completely passive, the corresponding substate of ϵ takes a zero value owing to a corresponding zero row in the matrix S as discussed in Remark 3, and $f(r)=1$ by (27). As a result, (30) and (32) indicate that a completely passive substate r is under the average consensus and finally converge to a consensus value. On the other hand, if there is at least one node i in the network that is active and has valid information on a substate r, $r=1,2,\ldots,n$ (i.e., there is at least one positive $k_i m_{ir}$ for $i=1,2,\ldots,N$), then $f(r)=0$ and (30) and (32) suggest the substate converge to the average of all valid active corresponding substates in the network.*

IV.Active-Passive Consensus Filters with Time-varying Information Node Roles

A. Proposed Architecture

In this section, we extend the intermediate result in Section III to the actual practical case, where both the active-passive role of each node and the local observer vectors $z_i(t)$, $i=1,2,\ldots,N$ are time-varying. However, owing to the properties of the overall time-varying system, we now consider that at any time moment t, for each substate of the process, there is at least one node that is active and has a valid information on that substate (that is, there is no completely passive substate at any time moment) as in Ref. 8. For this purpose, we consider the proposed active-passive consensus filter given by

$$\dot{x}_i(t) \quad = \quad -\alpha\sum_{i\sim j}(x_i(t)-x_j(t))+\alpha\sum_{i\sim j}(\xi_i(t)-\xi_j(t))-\alpha k_i(t)M_i(x_i(t)-z_i(t)),\ x_i(0)=x_{i0}, \tag{33}$$

$$\dot{\xi}_i(t) \quad = \quad -\gamma\Big(\sum_{i\sim j}(x_i(t)-x_j(t))+\sigma\xi_i(t)\Big),\ \xi_i(0)=\xi_{i0}, \tag{34}$$

where $x_i(t)\in\mathbb{R}^n$, $\xi_i(t)\in\mathbb{R}^n$, $z_i(t)\in\mathbb{R}^n$ denote the state, the integral action and the local estimate of node i, $i=1,\ldots,N$ respectively. M_i is the value of information matrix defined in (5). Moreover, $\alpha,\gamma\in\mathbb{R}_+$ are constant consensus gains. Note that $k_i(t)$ in this section is time-varying and $k_i(t)\in[0,1]$. We further assume that each node can smoothly change back and forth between active and passive role (i.e., $k_i(t)$ is a smooth function on the interval $[0,1]$). We also note again the discussion in Remark 2 here.

B. Stability Analysis

Let $x(t) \triangleq \begin{bmatrix}x_1^{\mathrm{T}}(t) & x_2^{\mathrm{T}}(t) & \ldots & x_N^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$, $\xi(t) \triangleq \begin{bmatrix}\xi_1^{\mathrm{T}}(t) & \xi_2^{\mathrm{T}}(t) & \ldots & \xi_N^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$, and $\zeta(t) \triangleq \begin{bmatrix}z_1^{\mathrm{T}}(t) & z_2^{\mathrm{T}}(t) & \ldots & z_N^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}$. Similar to (8) and (9), the proposed algorithm (33) and (34) can be rewritten in the compact form as

$$\dot{x}(t) \quad = \quad -\alpha F(t)x(t)+\gamma(\mathcal{L}(\mathcal{G})\otimes \mathrm{I}_n)\xi(t)+\alpha\mathcal{M}(t)\zeta(t),\quad x(0)=x_0, \tag{35}$$

$$\dot{\xi}(t) = -\gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)x(t) - \gamma\sigma\xi(t), \quad \xi(0) = \xi_0, \tag{36}$$

where $\mathcal{M}(t) \triangleq$ block–diag $\big(k_1(t)M_1, k_2(t)M_2, \ldots, k_N(t)M_N\big) \in \mathbb{R}^{Nn\times Nn}$, and $F(t) \triangleq \big(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n + \mathcal{M}(t)\big) \in \mathbb{R}^{Nn\times Nn}$. Since M_i for all $i = 1, 2, \ldots, N$ are diagonal matrices, $\mathcal{M}(t)$ is a diagonal matrix for any $t \geq 0$.

Similar to (11), we define

$$\begin{aligned} S(t) &\triangleq (\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}(t)(\mathbf{1}_N \otimes \mathrm{I}_n) \in \mathbb{R}^{n\times n} \\ &= k_1(t)M_1 + k_2(t)M_2 + \ldots + k_N(t)M_N, \end{aligned} \tag{37}$$

as a diagonal matrix that contains the total weight of all valid active substates of local observer vectors $z_i(t)$ on its diagonal. Note that since we assume at any time moment t, for each substate of the process, there is at least one node that is active and has a valid information on that substate, $S(t)$ is full rank as discussed in Remark 3; hence, it is invertible. We now let

$$\epsilon(t) \triangleq S^{-1}(t)(\mathbf{1}_N^{T} \otimes \mathrm{I}_n)\mathcal{M}(t)\zeta(t) \in \mathbb{R}^n \tag{38}$$

be the dynamic average of all valid active substates of local observer vectors $z_i(t)$, $i = 1, 2, \ldots, N$ in the network.

Next, we define the error as

$$\delta(t) \triangleq x(t) - (\mathbf{1}_N \otimes \epsilon(t)) \in \mathbb{R}^{Nn}, \tag{39}$$

$$e(t) \triangleq \xi(t) - \frac{\alpha}{\gamma}(\mathcal{L}^{+}(\mathcal{G}) \otimes \mathrm{I}_n)\mathcal{M}(t)\omega(t) \in \mathbb{R}^{Nn}, \tag{40}$$

where $\omega(t) \triangleq \big((\mathbf{1}_N \otimes \epsilon(t)) - \zeta(t)\big) \in \mathbb{R}^{Nn}$. Similar to (15) and (14), by taking the time derivative of (39) and (40), we obtain

$$\dot{\delta}(t) = -\alpha F(t)\delta(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)e(t) - \frac{\alpha}{N}(\mathbf{1}_N \otimes \mathrm{I}_n)(\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}(t)\omega(t) - (\mathbf{1}_N \otimes \dot{\epsilon}(t)), \;\; \delta(0) = \delta_0, \tag{41}$$

$$\begin{aligned} \dot{e}(t) = {} & -\gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\delta(t) - \gamma\sigma e(t) - \sigma\alpha(\mathcal{L}(\mathcal{G})^{+} \otimes \mathrm{I}_n)\mathcal{M}(t)\omega(t) \\ & -\frac{\alpha}{\gamma}(\mathcal{L}(\mathcal{G})^{+} \otimes \mathrm{I}_n)\big(\dot{\mathcal{M}}(t)\omega(t) + \mathcal{M}(t)\dot{\omega}(t)\big), \;\; e(0) = e_0. \end{aligned} \tag{42}$$

By Lemma 1 and Remark 4, we can further reduce (41) to

$$\dot{\delta}(t) = -\alpha F(t)\delta(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)e(t) - (\mathbf{1}_N \otimes \dot{\epsilon}(t)), \;\; \delta(0) = \delta_0. \tag{43}$$

Next, define

$$q_1(t) \triangleq -(\mathbf{1}_N \otimes \dot{\epsilon}(t)), \tag{44}$$

$$q_2(t) \triangleq -\sigma\alpha(\mathcal{L}(\mathcal{G})^{+} \otimes \mathrm{I}_n)\mathcal{M}(t)\omega(t) - \frac{\alpha}{\gamma}(\mathcal{L}(\mathcal{G})^{+} \otimes \mathrm{I}_n)\big(\dot{\mathcal{M}}(t)\omega(t) + \mathcal{M}(t)\dot{\omega}(t)\big). \tag{45}$$

Since $k_i(t)$ is a smooth function on the interval $[0, 1]$, $\mathcal{M}(t)$, $S(t)$, $S^{+}(t)$ and $\dot{\mathcal{M}}(t)$ are bounded. In addition, since $z_i(t)$ and $\dot{z}_i(t)$ are bounded by assumption, $\zeta(t)$ and $\dot{\zeta}(t)$ are bounded. Consequently, $\epsilon(t)$, $\omega(t)$, $\dot{\epsilon}(t)$, and $\dot{\omega}(t)$ are bounded. Therefore, $q_1(t)$ and $q_2(t)$ are bounded such as $\|q_1(t)\|_2 \leq q_1^*$ and $\|q_2(t)\|_2 \leq q_2^*$. (43) and (42) are now can be rewritten as

$$
\begin{aligned}
\dot{\delta}(t) &= -\alpha F(t)\delta(t) + \gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)e(t) + q_1(t),\ \delta(0) = \delta_0. && (46)\\
\dot{e}(t) &= -\gamma(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\delta(t) - \gamma\sigma e(t) + q_2(t),\ e(0) = e_0. && (47)
\end{aligned}
$$

Theorem 2. *Consider a sensor network with N nodes given by (33) and (34), where nodes exchange information using local measurements under a connected, undirected graph $\mathcal{G}$. Then, the closed-loop error dynamics (46) and (47) are uniformly ultimately bounded.*

Proof. Consider the Lyapunov function candidate given by (20). By taking time derivative of (20) along the trajectories of (46) and (47), we obtain

$$
\begin{aligned}
\dot{V}(\delta(t), e(t)) &= -\alpha\delta^{\mathrm{T}}(t)F(t)\delta(t) + \gamma\delta^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)e(t) + \delta(t)^{\mathrm{T}}q_1(t) - \gamma e^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\delta(t)\\
&\quad -\gamma\sigma e^{\mathrm{T}}(t)e(t) + e^{\mathrm{T}}(t)q_2(t)\\
&= -\alpha\delta^{\mathrm{T}}(t)F(t)\delta(t) - \gamma\sigma e^{\mathrm{T}}(t)e(t) + \delta^{\mathrm{T}}(t)q_1(t) + e^{\mathrm{T}}(t)q_2(t)\\
&\leq -\alpha\delta^{\mathrm{T}}(t)F(t)\delta(t) + \delta^{\mathrm{T}}(t)q_1(t) - \gamma\sigma\|e(t)\|_2^2 + \|e(t)\|_2 q_2^*\\
&\leq -\alpha\delta^{\mathrm{T}}(t)F(t)\delta(t) + \delta^{\mathrm{T}}(t)q_1(t) - \gamma\sigma\|e(t)\|_2\big(\|e(t)\|_2 - \phi_e\big), && (48)
\end{aligned}
$$

where $\phi_e \triangleq \frac{q_2^*}{\gamma\sigma}$. Let

$$
\mathcal{H} \triangleq -\alpha\delta^{\mathrm{T}}(t)F(t)\delta(t) + \delta^{\mathrm{T}}(t)q_1(t), \tag{49}
$$

and

$$
\psi(t) \triangleq J\delta(t) = J\begin{bmatrix}\delta_{11}(t)\\ \vdots\\ \delta_{1n}(t)\\ \vdots\\ \delta_{N1}(t)\\ \vdots\\ \delta_{Nn}(t)\end{bmatrix} = \begin{bmatrix}\left.\begin{matrix}\delta_{11}(t)\\ \vdots\\ \delta_{N1}(t)\end{matrix}\right\}\psi_1(t)\\ \vdots\\ \left.\begin{matrix}\delta_{1n}(t)\\ \vdots\\ \delta_{Nn}(t)\end{matrix}\right\}\psi_n(t)\end{bmatrix} = \begin{bmatrix}\psi_1(t)\\ \vdots\\ \psi_n(t)\end{bmatrix}, \tag{50}
$$

where J is the same permutation matrix discussed in subsection C and $\delta_{ij}(t)$ indicates the error of substate j, $j = 1, 2, \ldots, n$ of node i, $i = 1, 2, \ldots, N$. Note that

$$
\psi^{\mathrm{T}}(t)(JFJ^{\mathrm{T}})\psi(t) = \delta^{\mathrm{T}}(t)J^{\mathrm{T}}(JFJ^{\mathrm{T}})J\delta(t) = \delta^{\mathrm{T}}(t)F\delta(t). \tag{51}
$$

Utilize (51) and (44), (49) can be rewritten as

$$
\mathcal{H} = -\alpha\psi^{\mathrm{T}}(t)(JFJ^{\mathrm{T}})\psi(t) - \psi(t)J(\mathbf{1}_N \otimes \dot{\epsilon}(t)). \tag{52}
$$

Note also that

$$J(\mathbf{1}_N \otimes \dot{\epsilon}(t)) = J \begin{bmatrix} \dot{\epsilon}(t) \\ \vdots \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_N \dot{\epsilon}_1(t) \\ \vdots \\ \mathbf{1}_N \dot{\epsilon}_n(t) \end{bmatrix}, \tag{53}$$

where $\dot{\epsilon}_j(t)$ for $j = 1, 2, \ldots, n$ is the j-th substate of $\epsilon(t)$. By assuming that at any time moment t, for each substate of the process, there is at least one node that is active and has a valid information on that substate, the structure of $JF(t)J^{\mathrm{T}}$ in (26) shows us that for all $r = 1, 2, \ldots, n$, $F_r(t)$ can be rewritten as

$$F_r(t) = \begin{bmatrix} \mathcal{L}_{11} + k_1(t)m_{1r} & \cdots & \mathcal{L}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{N1} & \cdots & \mathcal{L}_{NN} + k_N(t)m_{Nr} \end{bmatrix} = \mathcal{L}(\mathcal{G}) + K_r(t) \tag{54}$$

where $K_r(t) = \mathrm{diag}(\begin{bmatrix} k_1(t)m_{1r} & \ldots & k_N(t)m_{Nr} \end{bmatrix}^{\mathrm{T}})$ with at least one of the element on the diagonal $\beta_i \triangleq k_i(t)m_{ir} \in \mathbb{R}_+$ for some $i \in \mathcal{V}_{\mathcal{G}}$. Thus, we can write $K_r(t) = K_{r0} + K_{r1}(t)$ where $K_{r0} \triangleq \mathrm{diag}(\begin{bmatrix} 0 & \ldots & 0 & \beta_i & 0 & \ldots & 0 \end{bmatrix}^{\mathrm{T}})$, and $K_{r1} \triangleq K_r(t) - K_{r0}$ is also a diagonal matrix with nonnegative elements on the diagonal. As a result, we have

$$F_r(t) = \mathcal{L}(\mathcal{G}) + K_{r0} + K_{r1}(t) = F_{r0} + K_{r1}(t), \tag{55}$$

where $F_{r0} \triangleq \mathcal{L}(\mathcal{G}) + K_{r0}$ is a positive definite matrix by, for example, Lemma 2 in Ref. 13 or Lemma 3.3 in Ref. 14. We now can write

$$JF(t)J^{\mathrm{T}} = F_0 + \mathcal{M}_0(t), \tag{56}$$

where $F_0 \triangleq \text{block–diag}(F_{10}, F_{20}, \ldots, F_{n0})$ is a positive definite matrix and $\mathcal{M}_0(t) \triangleq \text{block–diag}(K_{11}(t), K_{21}(t), \ldots, K_{n1}(t))$ is a diagonal matrix with nonnegative elements on the diagonal. From (56), (52) now becomes

$$\begin{aligned} \mathcal{H} &= -\alpha\psi^{\mathrm{T}}(t)\big(F_0 + \mathcal{M}_0(t)\big)\psi(t) - \psi(t)J(\mathbf{1}_N \otimes \dot{\epsilon}(t)) \\ &\le -\alpha\psi^{\mathrm{T}}(t)F_0\psi(t) - \psi(t)J(\mathbf{1}_N \otimes \dot{\epsilon}(t)) \\ &\le -\alpha\lambda_{\min}(F_0)\|\psi(t)\|_2^2 + \|\psi(t)\|_2 q_1^* \\ &\le -\alpha\lambda_{\min}(F_0)\|\psi(t)\|_2(\|\psi(t)\|_2 - \phi_\delta), \end{aligned} \tag{57}$$

where $\phi_\delta = \frac{q_1^*}{\alpha\lambda_{\min}(F_0)}$ with $\|J(\mathbf{1}_N \otimes \dot{\epsilon}(t))\|_2 = \|(\mathbf{1}_N \otimes \dot{\epsilon}(t))\|_2 \le q_1^*$. We note here again that $\psi(t) \triangleq J\delta(t)$; that is, $\psi(t)$ is a permutation of $\delta(t)$, and since a vector norm is preserved under permutation, $\|\psi(t)\|_2 = \|\delta(t)\|_2$. Therefore, $\mathcal{H} \le 0$ outside the compact set $\Omega_\delta \triangleq \{\big(\delta(t), e(t)\big) : \|\psi(t)\|_2 \le \phi_\delta\} = \{\big(\delta(t), e(t)\big) : \|\delta(t)\|_2 \le \phi_\delta\}$.

By combining the result of (48) and (57), we have $\dot{V}(\delta(t), e(t)) \leq 0$ outside the compact set given by

$$\Omega \triangleq \{(\delta(t), e(t)) : \|\delta(t)\|_2 \leq \phi_\delta\} \cap \{(\delta(t), e(t)) : \|e(t)\|_2 \leq \phi_e\}. \tag{58}$$

Consequently, the closed-loop error dynamics given by (46) and (47) are uniformly bounded. ■

The following corollary to the above theorem is immediate.

Corollary 1. *Consider a sensor network with N nodes given by (33) and (34), where nodes exchange information using local measurements under a connected, undirected graph $\mathcal{G}$. Then, the ultimately bound of $\delta(t)$ for $t \geq T$ is given by*

$$\|\delta(t)\|_2^2 \leq \frac{(q_1^*)^2}{\alpha^2 \lambda_{\min}(F_0)^2} + \frac{(q_2^*)^2}{\gamma^2 \sigma^2}, \tag{59}$$

where q_1^* and q_2^* are the upper bounds of $\|q_1(t)\|_2^2$ and $\|q_2(t)\|_2^2$ defined in (44) and (45).

Proof. From the proof of Theorem 2, we have $\dot{V}(\delta(t), e(t)) \leq 0$ outside the compact set Ω given by (58). Therefore, the evolution of $V(\delta(t), e(t))$ is upper bounded by

$$V(\delta(t), e(t)) \leq \max_{(\delta(t), e(t)) \in \Omega} V(\delta(t), e(t)) = \frac{1}{2}(\phi_\delta^2 + \phi_e^2). \tag{60}$$

Note that $\frac{1}{2}\delta^{\mathrm{T}}(t)\delta(t) \leq V(\delta(t), e(t))$, thus (59) is immediate ■

Remark 6. *It should be noted that by definition of $q_2(t)$ in (45), the upper bound q_2^* can be rewritten as $q_2^* = \alpha q_3^*$. As a result, Corollary 1 indicates if the gains α, γ, and σ are chosen properly such that $\frac{1}{\alpha^2}$ and $\frac{\alpha^2}{\gamma^2\sigma^2}$ are small, then the ultimate bound (59) of $\delta(t)$ is small when $t \geq T$; and hence, the overall performance of the sensor network can be improved.*

V. Information Validity Monitor Layer

In this section, we present a dynamic average consensus that is parallel to the active-passive consensus filter in order to monitor the validity of the information (see Figure 1). For this purpose, consider the dynamics given by

$$\dot{q}_i(t) = -\beta \sum_{i \sim j} \big(q_i(t) - q_j(t)\big) + \beta \sum_{i \sim j} \big(r_i(t) - r_j(t)\big) - \beta\big(q_i(t) - h_i(t)\big), \quad q_i(0) = q_{i0}, \tag{61}$$

$$\dot{r}_i(t) = -\mu \sum_{i \sim j} \big(q_i(t) - q_j(t)\big), \quad r_i(0) = r_{i0}, \tag{62}$$

where $q_i(t) \in \mathbb{R}^n$ denotes the information validity vector for node i, $i = 1, 2, \ldots, N$, $h_i(t) \triangleq k_i(t) m_i \in \mathbb{R}^n$ with m_i is the diagonal of the value of information matrix M_i, and $\beta, \mu \in \mathbb{R}_+$ denote the gains.

Note that the structure of (61) and (62) is a special case of (6) and (7), where it becomes a dynamic average consensus, for which a proof can be found in, for example Refs. 1 and 15 and references therein. Therefore, $q_i(t)$ is tracking the neighborhood of the dynamic average $\bar{h}(t) \triangleq \frac{\mathbf{1}\mathbf{1}^{\mathrm{T}}}{N} h(t) \in \mathbb{R}^n$ with $h(t) =$

$\begin{bmatrix} h_1^{\mathrm{T}}(t) & h_2^{\mathrm{T}}(t) & \ldots & h_N^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$. Since $\bar{h}(t)$ is the dynamic average of the $k_i(t)m_i$ for all $i = 1, 2, \ldots, N$, $q_i(t)$ provides us the information on the average of active weights of the whole network at each time moment for each substate of $z_i(t)$, $i = 1, 2, \ldots, N$. Therefore, $q_i(t)$ can be considered as a confidence factor to check the reliability of the information. For example, the value of a substate of $q_i(t)$ increases as the number of valid active corresponding substates of $z_i(t)$ for all $i = 1, 2, \ldots, N$ in the whole network increases; that is, the higher the value of $q_i(t)$ is, the more reliable the information is. This point would become more apparent as illustrated in examples of Section VI.

VI. Discussion and Examples

In this section, we present several numerical examples to illustrate the results given in previous section. For this purpose, consider a process composed of two subprocesses with the dynamics given by (1), where

$$A = \text{block–diag}\left(A_1, A_2\right) = \begin{bmatrix} 0.0150 & 0 & 0 & 0 \\ 0 & -0.0250 & 0 & 0 \\ 0 & 0 & -0.0005 & 0.1000 \\ 0 & 0 & -0.2500 & 0 \end{bmatrix}. \tag{63}$$

with $A_1 \triangleq \begin{bmatrix} 0.0150 & 0 \\ 0 & -0.0250 \end{bmatrix}$ and $A_2 \triangleq \begin{bmatrix} -0.0005 & 0.1000 \\ -0.2500 & 0 \end{bmatrix}$. We consider a sensor network with 4 nodes exchange information among each other using their local measurements according to a connected, undirected ring graph. Each node's sensing capability is represented by (3) with the output matrices

$$^{I}C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{64}$$

$$^{III}C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \tag{65}$$

$$^{I}C_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \tag{66}$$

$$^{I}C_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \tag{67}$$

and hence, for the local observers ${}^1\bar{C}_1 = [1\ 0]$, ${}^1\bar{C}_2 = [0\ 1]$ corresponding to A_1 and ${}^2\bar{C}_3 = [1\ 0]$, ${}^2\bar{C}_4 = [0\ 1]$ corresponding to A_2. Note that sensors 1, 3, and 4 are Category I sensors since the pair $(A_1, {}^1\bar{C}_1)$ is detectable, and the pairs $(A_2, {}^2\bar{C}_3)$ and $(A_2, {}^2\bar{C}_4)$ are observable. On the other hand, the pair $(A_1, {}^1\bar{C}_2)$ is unobservable, so sensor 2 is a Category III sensor. As a result, the observer structure (4) is only applied to sensors 1, 3, and 4. For example,

$$\dot{s}_1(t) = A_1 s_1(t) + {}^1L_1(y_1 - {}^1\bar{C}_1 s_1(t)), \tag{68}$$

where $s_1(t) \in \mathbb{R}^2$ and ${}^1L_1 = \begin{bmatrix} 2.0302 \\ 0 \end{bmatrix}$. Similarly, for sensors 3 and 4, $s_3(t), s_4(t) \in \mathbb{R}^2$ and we choose

$$
\begin{aligned}
{}^2L_3 &= \begin{bmatrix} 5.3737 \\ 3.9039 \end{bmatrix}, && (69) \\
{}^2L_4 &= \begin{bmatrix} -1.8052 \\ 2.4094 \end{bmatrix}. && (70)
\end{aligned}
$$

As discussed in Section II.B, after using the local observers to estimate the states of the subprocesses, we are now able to construct $z_i(t) \in \mathbb{R}^4$ such that

$$
\begin{aligned}
z_1(t) &= \begin{bmatrix} s_1^{\mathrm{T}}(t) & 0 & 0 \end{bmatrix}^{\mathrm{T}}, && (71) \\
z_2(t) &= \begin{bmatrix} 0 & y_2(t) & 0 & 0 \end{bmatrix}^{\mathrm{T}}, && (72) \\
z_3(t) &= \begin{bmatrix} 0 & 0 & s_3^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, && (73) \\
z_4(t) &= \begin{bmatrix} 0 & 0 & s_4^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}. && (74)
\end{aligned}
$$

Next, we define the value of information matrix for each node. Since the pair $(A_1, {}^1\bar{C}_1)$ is detectable, sensor 1 can observe the first substate of the process, yet the estimation of second substate of the process is not so well. Thus, we can choose $M_1 = \mathrm{diag}\left(\begin{bmatrix} 2 & 0.5 & 0 & 0 \end{bmatrix}\right)$. In the same manner, we choose $M_2 = \mathrm{diag}\left(\begin{bmatrix} 0 & 2 & 0 & 0 \end{bmatrix}\right)$, $M_3 = \mathrm{diag}\left(\begin{bmatrix} 0 & 0 & 2 & 1 \end{bmatrix}\right)$ and $M_4 = \mathrm{diag}\left(\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}\right)$.

In addition, all nodes are subject to random initial conditions and we set $\alpha = 15$, $\gamma = 10$, $\sigma = 0.1$, $\beta = 10$ and $\mu = 5$. Furthermore, for the transition of $k_i(t)$, we use the function $k_i(t) = e^{-\theta t}$ when node i is switching from 1 to 0, and $k_i(t) = 1 - e^{-\theta t}$ when node i is switching from 0 to 1, where $\theta = 5$. The simulations are run for 100 second. All below examples share the same setup but are different in node roles over time.

Example 1. In this example, the role of each node are fixed over time. Specifically, nodes 1, 2, and 3 are active (i.e, $k_1(t) = k_2(t) = k_3(t) = 1$) while node 4 is passive (i.e, $k_4(t) = 0$). Note that this information node role setup satisfies the assumption in Section IV that for each process's substate, there is at least one node that is active and has a valid information on that substate. Figure 2 shows the performance of the sensor networks under the proposed active-passive consensus filter given by (33) and (34), where nodes quickly converge to consensus and are able to closely estimate the process states. In addition, Figure 3 shows the result of the information validity monitor layer given by (61) and (62). Since the information node role setup in this example is fixed, the information validity vectors converge to constants. As can be seen, for example, in the bottom plot of Figure 3, the information validity vector of the process' fourth substate converges to a low value due to the fact that node 4, which can directly senses this substate, is passive and the information in this substate is obtained only from the local observer of node 3.

Example 2. In this example, the role of each node varies over time. For the first 25 seconds, nodes 1 and 4 are active (i.e, $k_1(t) = k_4(t) = 1$ and $k_2(t) = k_3(t) = 0$); for $t \in (25, 50]$, nodes 1 and 3 are active; for $t \in (50, 75]$, all 4 nodes are active; for $t \in (75, 100]$, nodes 1, 2 and 3 are active. This information node

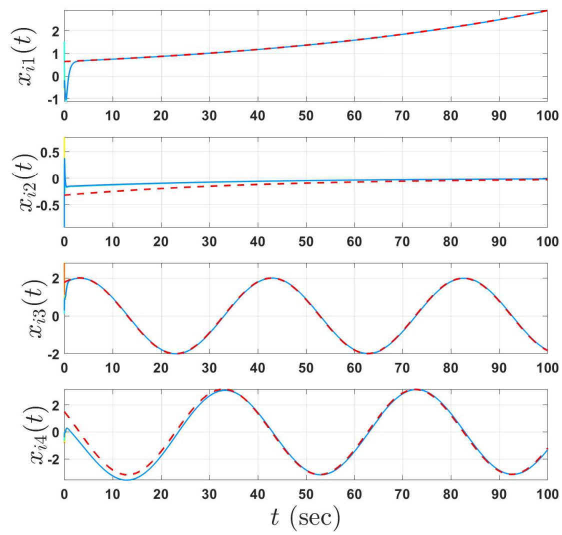

Figure 2. State estimates of the sensor network with four nodes in a connected, undirected ring graph in Example 1 under the proposed active-passive consensus filter architecture (33) and (34) (the dash lines denote the states of the actual process and the solid lines denote the state estimates of nodes).

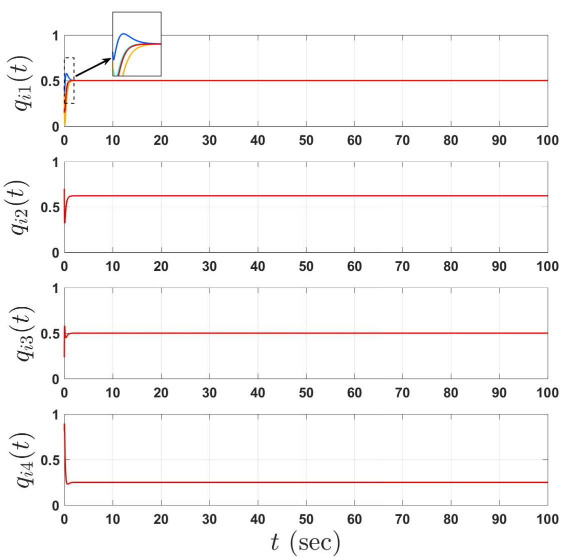

Figure 3. The evolution of information validity vector of the sensor network with four nodes in a connected, undirected ring graph in Example 1 under the monitor layer given by (61) and (62).

role configuration satisfies the assumption in Section IV that for each process's substate, there is at least one node that is active and has a valid information on that substate. Figure 4 shows the performance of the sensor networks under the proposed active-passive consensus filter given by (33) and (34), where nodes quickly converge to consensus and are able to closely estimate the process states. In addition, Figure 5 shows the result of the information validity monitor layer given by (61) and (62). Since the information node roles change with respect to time in this example, the information validity vectors converge to different values over time. For example, for the first 50 seconds, the value of the information validity vectors $q_{i2}(t)$ on the second

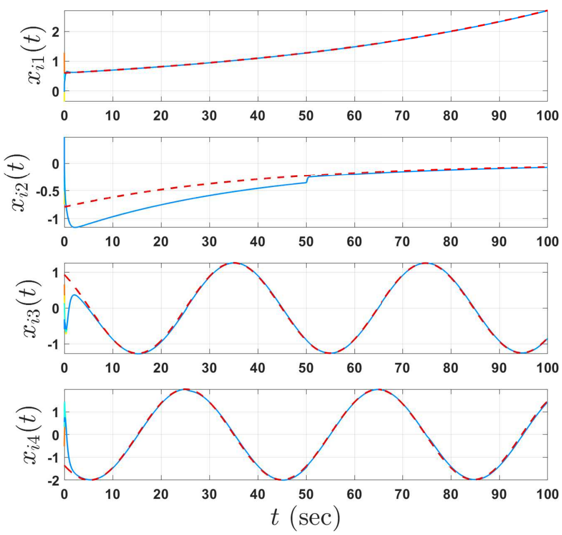

Figure 4. State estimates of the sensor network with four nodes in a connected, undirected ring graph in Example 2 under the proposed active-passive consensus filter architecture (33) and (34) (the dash lines denote the states of the actual process and the solid lines denote the state estimates of nodes).

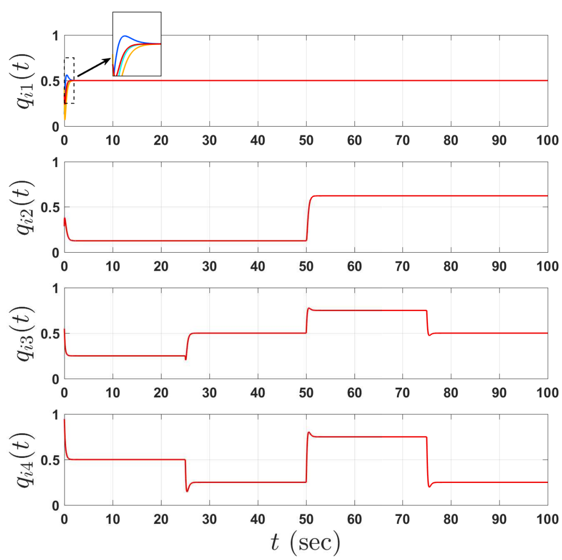

Figure 5. The evolution of information validity vector of the sensor network with four nodes in a connected, undirected ring graph in Example 2 under the monitor layer given by (61) and (62).

substate of the process is small since the information on this substate is obtained from the local observer of node 1, which has the weight of 0.5 only. For the last 50 seconds, the value of $q_{i2}(t)$ has increased since node 2 becomes active and adds more validity on the information.

Example 3. In this example, the role of each node varies over time such that for the first 25 second, nodes 2 and 4 are active (i.e, $k_2(t) = k_4(t) = 1$ and $k_1(t) = k_3(t) = 0$); for $t \in (25, 50]$, nodes 3 and 4 are active; for $t \in (50, 75]$, all 4 nodes are active; for $t \in (75, 100]$, nodes 1 and 2 are active. This information node role configuration indeed violates the assumption in Section IV that for each process's substate, there is at least one node that is active and has a valid information on that substate. In fact, this configuration allows some substates becomes completely passive at some time instants, and it can be another important practical case. For example, if the process of interest represents multiple targets with each subprocess corresponding to a target, then at any time instant, a target can or can not be observed by the sensor network.

Figure 6 shows the performance of the sensor networks under the proposed active-passive consensus filter given by (33) and (34), where nodes quickly converge to consensus and are able to closely estimate the actual value if that particular process substate can be observed by at least one node (i.e., at least one node is active and has valid information on that substate). By utilizing the information validity monitor layer given by (61) and (62), one can monitor if the information of a substate is valid or not (that is, if $q_{ij}(t) = 0$, then the substate j is completely passive, and thus the information is invalid and not reliable) as shown in Figure 7. It can be seen that, for example, during the first 50 seconds the first substate of the process is not observable (i.e., completely passive), and hence the information from the sensors on this substate is not valid. Another example is that during the last 25 seconds, the third and fourth substates of the process are completely passive (see Figure 7) and the corresponding substates obtained from the sensor network are constants during this time period as seen in Figure 6. As discussed in Remark 5 of Section III, completely passive substates still result in nodes reaching consensus, yet the information is invalid. Note that at time $t = 75$ seconds, the nodes' estimates already reached the consensus, thus when these substates becomes completely passive, the sensors retains the last values they sense from the process until at least one of the sensor becomes active and has valid information on these substates.

Intuitively, from the analysis in Section III, we expect that when extending the architecture (6) and (7) to the time-varying case, that is, the proposed active-passive consensus filter in Section IV given by (33) and (34), the sensor networks can converge to the null space of $F(t)$. However, without the assumption in Section IV (that is, for each process's substate, there is at least one node that is active and has a valid information on that substate), at each time instant $F(t)$ can be either positive definite or nonnegative definite, and hence, the null space of $F(t)$ is time-varying as well. Utilizing Lyapunov analysis in this case is a good and challenging future research direction to the authors.

VII. Conclusion

This paper contributed to the previous studies in heterogeneous sensor networks through proposing a dynamic information fusion framework for sensor networks with the integration of local observers, value of information, active-passive consensus filters, and a layer to monitor the validity of information. The proposed framework considered a process of interest with multiple subprocesses and the sensor network that allows nodes with heterogeneous modalities, heterogeneous information node roles, and heterogeneous quality of information. In addition, the extra layer allows operators to evaluate the reliability of the fused information based on the local feedbacks received from the sensor network. In addition to the presented theoretical algorithms, illustrative examples had shown the efficacy of the proposed structure and prompted a discussion on the practical aspects when relaxing some certain assumptions.

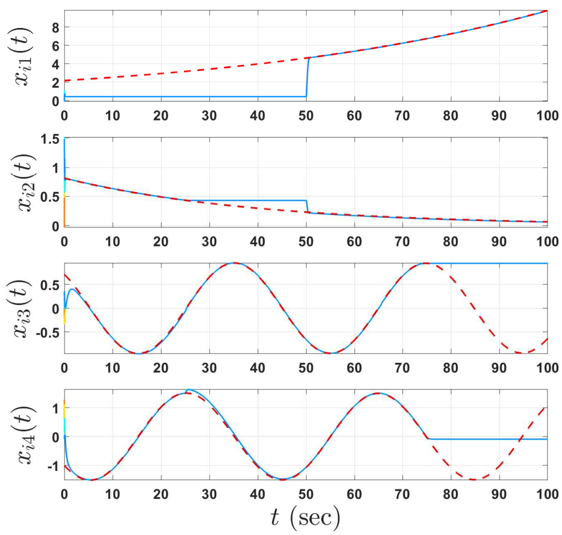

Figure 6. State estimates of the sensor network with four nodes in a connected, undirected ring graph in Example 3 under the proposed active-passive consensus filter architecture (33) and (34) (the dash lines denote the states of the actual process and the solid lines denote the state estimates of nodes).

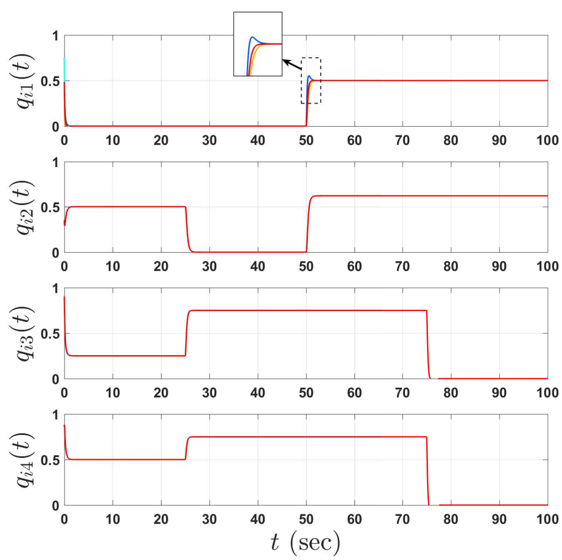

Figure 7. The evolution of information validity vector of the sensor network with four nodes in a connected, undirected ring graph in Example 3 under the monitor layer given by (61) and (62).

Appendix

A. Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{Z}_+$ denotes the set of positive integer numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+^{n\times n}$ (resp., $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative definite) real matrices, $\mathbf{1}_n$ denotes the $n \times 1$ vector of all ones, and I_n denotes the $n \times n$ identity matrix. In addition, we write $(\cdot)^{\mathrm{T}}$ for transpose, $(\cdot)^{+}$ for generalized inverse, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ for the minimum and maximum eigenvalue of the symmetric matrix A, respectively, $\lambda_i(A)$ for the i-th eigenvalue of A, where A is symmetric and the eigenvalues are ordered from least to greatest value, block−diag$(A_1, \ldots, A_n)$ for the block diagonal matrix with $A_1, \ldots, A_n$ are square matrices lying along the diagonal and all other entries of the matrix equal 0, diag(a) for the diagonal matrix with the vector a on its diagonal, $[x]_i$ for the entry of the vector x on the i-th row, and A_{ij} for the entry of the matrix A on the i-th row and j-th column. In addition, for $A \in \mathbb{R}^{n\times m}$, $\mathcal{R}(A)$ denotes the range of A, rank(A) denotes the rank of A, $\mathcal{N}(A)$ denotes the null space of A, def$(A) \triangleq \dim \mathcal{N}(A)$ denotes the defect of A.

Next, we recall some basic notions from graph theory and refer to textbooks Refs. 16 and 17 for details. Specifically, an undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_{\mathcal{G}} = \{1, \ldots, N\}$ of nodes and a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of edges. If $(i, j) \in \mathcal{E}_{\mathcal{G}}$, then the nodes i and j are neighbors and the neighboring relation is indicated with $i \sim j$. The degree of a node is given by the number of its neighbors. Letting d_i be the degree of node i, then the degree matrix of a graph $\mathcal{G}$, $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N\times N}$, is given by $\mathcal{D}(\mathcal{G}) \triangleq \mathrm{diag}(d)$, $d = [d_1, \ldots, d_N]^{\mathrm{T}}$. A path $i_0 i_1 \ldots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $\mathcal{G}$ is connected if there is a path between any pair of distinct nodes. The adjacency matrix of a graph $\mathcal{G}$, $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N\times N}$, is given by $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ if $(i, j) \in \mathcal{E}_{\mathcal{G}}$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The Laplacian matrix of a graph, $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{R}}_{+}^{N\times N}$, playing a central role in many graph-theoretic treatments of sensor networks, is given by $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. The spectrum of the Laplacian of an undirected and connected graph can be ordered as $0 = \lambda_1(\mathcal{L}(\mathcal{G})) < \lambda_2(\mathcal{L}(\mathcal{G})) \leq \cdots \leq \lambda_N(\mathcal{L}(\mathcal{G}))$ with $\mathbf{1}_N$ as the eigenvector corresponding to the zero eigenvalue $\lambda_1(\mathcal{L}(\mathcal{G}))$ and $\mathcal{L}(\mathcal{G})\mathbf{1}_N = \mathbf{0}_N$ and $\mathrm{e}^{\mathcal{L}(\mathcal{G})}\mathbf{1}_N = \mathbf{1}_N$. Here, we assume that the graph $\mathcal{G}$ of a given sensor network is undirected and connected.

The following lemmas are necessary for the main results of this paper.

Lemma A.1 [Lemma 3, 18]. *The Laplacian of a connected, undirected graph satisfies* $\mathcal{L}(\mathcal{G})\mathcal{L}^{+}(\mathcal{G}) = \mathrm{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$

Lemma A.2 [Fact 2.10.12, 19]. *Let* $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{m\times l}$. *Then, rank*$(AB) =$ *rank*(A) *if and only if* $\mathcal{R}(AB) = \mathcal{R}(A)$.

Lemma A.3 [Fact 6.4.43, 19]. *Let* $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{n\times l}$. *Then,* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ *if and only if* $BB^{+}A = A$.

Lemma A.4 [Theorem 2.4.3, 19]. *Let* $A \in \mathbb{R}^{n\times m}$, *then* $\mathcal{N}(A) = \mathcal{N}(A^{\mathrm{T}}A)$.

B. Proof of Lemma 1

Let $H \triangleq (\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)$, then (12) becomes

$$\epsilon = S^{+}H\mathcal{M}\zeta = (H\mathcal{M}H^{\mathrm{T}})^{+}H\mathcal{M}\zeta. \tag{75}$$

It should be noted that

$$(\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}(\mathbf{1}_N \otimes \epsilon) = H\mathcal{M}(\mathbf{1}_N \otimes \epsilon) = S\epsilon. \tag{76}$$

Utilize (75) and (76), the left hand side of (16) can be rewritten as

$$\begin{aligned}(\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}\omega &= (\mathbf{1}_N^{\mathrm{T}} \otimes \mathrm{I}_n)\mathcal{M}\big((\mathbf{1}_N \otimes \epsilon) - \zeta\big) \\ &= H\mathcal{M}(\mathbf{1}_N \otimes \epsilon) - H\mathcal{M}\zeta \\ &= S\epsilon - H\mathcal{M}\zeta \\ &= (H\mathcal{M}H^{\mathrm{T}})(H\mathcal{M}H^{\mathrm{T}})^{+}H\mathcal{M}\zeta - H\mathcal{M}\zeta \\ &= \big((H\mathcal{M}H^{\mathrm{T}})(H\mathcal{M}H^{\mathrm{T}})^{+}H\mathcal{M} - H\mathcal{M}\big)\zeta \\ &= R\zeta, \end{aligned} \tag{77}$$

where $R \triangleq \big((H\mathcal{M}H^{\mathrm{T}})(H\mathcal{M}H^{\mathrm{T}})^{+}H\mathcal{M} - H\mathcal{M}\big)$.

The matrix $H\mathcal{M}$ can be rewritten as

$$\begin{aligned} H\mathcal{M} &= \begin{bmatrix} k_1M_1 & k_2M_2 & \ldots & k_NM_N \end{bmatrix} \\ &= \begin{bmatrix} k_1\,\mathrm{diag}(m_1) & k_2\,\mathrm{diag}(m_2) & \ldots & k_N\,\mathrm{diag}(m_N) \end{bmatrix} \in \mathbb{R}^{n\times Nn}. \end{aligned} \tag{78}$$

We now define $\bar{m} \triangleq k_1m_1 + k_2m_2 + \ldots + k_Nm_N$, and note that $N > 1$. Clearly, $\mathrm{rank}(H\mathcal{M}) \leq n$. In addition, since elements of m_i for $i = 1, \ldots, N$ are nonnegative and the column vectors of $H\mathcal{M}$ are multiples of $e_1, e_2, \ldots, e_n$ where e_j is the unit vector with the j-th element is 1 and 0 otherwise, $\bar{m}$ only obtains an 0 element when $H\mathcal{M}$ has a zero row. Therefore,

$$\begin{aligned} \mathrm{rank}(H\mathcal{M}) &= \text{number of positive elements in } \bar{m} \\ &= n - (\text{number of 0 elements in } \bar{m}). \end{aligned} \tag{79}$$

Similarly, S can be rewritten as

$$\begin{aligned} S &= \mathrm{diag}(k_1m_1 + k_2m_2 + \ldots + k_Nm_N) \\ &= \mathrm{diag}(\bar{m}). \end{aligned} \tag{80}$$

Hence, it follows directly that

$$\begin{aligned} \mathrm{rank}(S) &= \text{number of positive elements in } \bar{m} \\ &= n - (\text{number of 0 elements in } \bar{m}). \end{aligned} \tag{81}$$

From (79) and (81), we have

$$\mathrm{rank}(H\mathcal{M}) = \mathrm{rank}(S) = \mathrm{rank}(H\mathcal{M}H^{\mathrm{T}}). \tag{82}$$

Utilize (82) and the result of Lemma A.2 with $A \triangleq H\mathcal{M}$ and $B \triangleq H^{\mathrm{T}}$, we obtain

$$\mathcal{R}(H\mathcal{M}) = \mathcal{R}(S) = \mathcal{R}(H\mathcal{M}H^{\mathrm{T}}). \tag{83}$$

Therefore, it now follows directly from Lemma A.3 that

$$(H\mathcal{M}H^{\mathrm{T}})(H\mathcal{M}H^{\mathrm{T}})^{+}H\mathcal{M} = H\mathcal{M}, \tag{84}$$

or $R = 0$. As a result, (16) is now immediate. ■

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