Finite-Time Control of Perturbed Dynamical Systems Based on a Generalized Time Transformation Approach*

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Abstract

We study finite-time control of perturbed dynamical systems based on the time transformation approach. For addressing time-critical applications, where the execution of a control algorithm over a prescribed time interval $[0, \tau)$ is necessary with τ being a user-defined convergence time, we introduce a new class of scalar, time-varying gain functions entitled as "generalized finite-time gain functions" that have the capability to convert an original baseline control algorithm into a time-varying one. Based on these generalized finite-time gain functions, in particular, the corresponding "generalized time transformation functions" are obtained and used to transform a resulting algorithm over the prescribed time interval $[0, \tau)$ to an equivalent algorithm over the stretched infinite-time interval $[0, \infty)$ for stability analysis, where the connection between the generalized finite-time gain functions and their corresponding generalized time transformation functions are investigated in detail. A procedure for designing finite-time control algorithms is further proposed and illustrated by numerical examples showing that the method is applicable to, but not limited to, a class of nonlinear systems as well as multiagent systems. In addition, we show all the conditions on the proposed generalized finite-time gain functions that guarantee the boundedness and convergence of the state and control signals. An application

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of our theoretical findings to the distributed control of networked multiagent systems problem over a prescribed time interval is also presented.

Keywords: Finite-time control; Time transformation; Multiagent systems

1. Introduction

In many practical applications such as engagement of a guided missile with a target, landing of an aerial vehicle at a non-stationary carrier, and sequential execution of given complex tasks, finite-time control algorithms play an important role (see, for example, [1, 2, 3, 4] and the references therein). These time-critical applications are often performed over a time interval $[0, \tau)$, where the utilized finite-time control algorithms are expected to guarantee a task completion at a user-defined convergence time τ.

1.1. Literature Review

Lyapunov theory for finite-time stabilization and corresponding non-Lipschitzian control methods are introduced in [1, 2, 5]. Since then, such tools have been utilized for addressing a wide range of problems in various applications. Specifically, in multiagent systems, non-Lipschitzian finite-time control algorithms are developed to solve consensus problems [6, 7, 8, 9, 10, 11, 12], formation tracking problems [13, 14], and containment problems [15, 16], to name but a few examples. Since these methods aim to develop time-invariant controllers, the convergence time depends on initial conditions; hence, τ may not be assigned by a control designer.

To provide a remedy to this problem, several results (see, for example, [17, 18, 19, 20, 21, 22, 23] and references therein) focus on developing control algorithms with bounded convergence time regardless of initial conditions (fixed-time convergence). A representative result is presented in [24], where the authors utilize homogeneous approximation and show that under some conditions, if the degree of the homogeneous approximation in 0-limit is strictly negative while the degree of the homogeneous approximation in ∞-limit is strictly positive, then the convergence time does not depend on the initial condition. Yet, the calculated upper bounds for the convergence times do not necessarily hold globally and they can be conservative. A recent result presented in [25] also provides the necessary and sufficient conditions for the fixed-time stability. In addition, there are studies that allow a user-defined convergence time τ to be assigned to the finite-time algorithms utilized in time-critical applications. For example, the authors of [26] propose a class of distributed control protocols to solve the consensus problem of linear multiagent systems within a prescribed time by reducing the sampling time as time progresses. As another example, the authors of [27] propose a methodology for designing autonomous and non-autonomous pre-defined settling time systems. However, their results still require some knowledge of initial conditions. Furthermore, a group of papers such as [3, 4, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] shares the same idea of introducing a time-varying gain into the controller for solving the control problem in a user-defined convergence time. Yet, different techniques for stability analysis are utilized in these papers. For instance, while [31, 32, 33, 34, 35, 36] focus on showing the stability of the system via the explicit solution, the authors of [37, 38, 39] utilize the comparison principle (see, for example, section 3.4 of [40]) on the Lyapunov function candidate for stability analysis. Taking another path, the authors of [3, 4] utilize a novel time transformation approach to transform the original system into an equivalent system on a stretched time interval $[0, \infty)$ and analyze this transformed system. The results of this paper are particularly related to and generalize the recent studies in [3, 4].

1.2. Contribution

In this paper, we study finite-time control of perturbed dynamical systems based on the time transformation approach. For addressing time-critical applications, where the execution of a control algorithm over a prescribed time interval $[0, \tau)$ is necessary with τ being a user-defined convergence time, we introduce a new class of scalar, time-varying gain functions entitled "generalized finite-time gain functions" that have the capability to convert an original baseline control algorithm into a time-varying one. Based on these generalized finite-time gain functions, in particular, the corresponding "generalized time transformation functions" are obtained and used to transform the resulting closed-loop system over the prescribed time interval $[0, \tau)$ into an equivalent one over the stretched infinite-time interval $[0, \infty)$ for stability analysis. The contributions of this paper are now stated:

- The time transformation method is introduced for allowing tools and methods from standard Lyapunov analysis to be used for analyzing the stability of the transformed system on the stretched time interval and then drawing a conclusion for the original system.
- The finite-time gain function and its corresponding time transformation function are generalized and their relationship is characterized.
- A procedure is further proposed for designing finite-time control algorithms together with illustrative numerical examples showing that the method is applicable to, but not limited to, a class of nonlinear systems as well as multiagent systems.
- System-theoretical conditions for guaranteeing the boundedness of control signals are also investigated in detail.

1.3. Organization

The paper is organized as follows. In Section 2, we state the necessary mathematical preliminaries and a key lemma for our main results. The proposed generalized time transformation functions-based finite-time control problem over the prescribed time interval $[0, \tau)$ is introduced and analyzed in Section 3. We also present an application of our theoretical findings to the distributed control of networked multiagent systems problem over a prescribed time interval in Section 4. Finally, our concluding remarks are summarized in Section 5. Note that a preliminary conference version of this paper is appeared in [41]. The present paper considerably expands on [41] by providing the detailed proofs of all the results together with additional remarks and discussions.

2. Mathematical Preliminaries

Standard mathematical notations are used in this paper. We write $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively for the minimum and maximum eigenvalue of a matrix A, $\mathrm{Re}(\lambda_{\min}(A))$ and $\mathrm{Re}(\lambda_{\max}(A))$ respectively for the real part of the minimum and maximum eigenvalue of a matrix A, $\lambda_i(A)$ for the i-th eigenvalue of A, where A is symmetric and the eigenvalues are ordered from least to greatest value, $\mathrm{diag}(a)$ for the diagonal matrix with the vector a on its diagonal, and A_{ij} for the entry of the matrix A on the i-th row and j-th column.

We next summarize the basic graph-theoretical notions used in this paper (see, for example, [42] and [43] for details). In particular, an undirected graph $\mathcal{G}$ is defined by a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of edges and a set $\mathcal{V}_{\mathcal{G}} = \{1, \ldots, N\}$ of nodes. We utilize $(i,j) \in \mathcal{E}_{\mathcal{G}}$ for cases when a pair of nodes i and j are neighbors, where $i \sim j$ indicates the neighboring relation. Furthermore, the degree of a node is determined by the number of its neighbors. Denoting d_i as the degree of node i, the degree matrix of a graph $\mathcal{G}$, $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N\times N}$, is defined by $\mathcal{D}(\mathcal{G}) \triangleq \mathrm{diag}(d)$, $d = [d_1, \ldots, d_N]^{\mathrm{T}}$. A path $i_0 i_1 \ldots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $\mathcal{G}$ is said to be connected when there is a path between any pair of distinct nodes. The adjacency matrix of a graph $\mathcal{G}$, $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N\times N}$, is defined by $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $(i,j) \in \mathcal{E}_{\mathcal{G}}$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The Laplacian matrix of a graph, $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{R}}_{+}^{N\times N}$, is then defined by $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$.

Finally, the following key lemma from [40, Theorem 4.14] is necessary for the results in this paper.

Lemma 1. *For a given dynamical system $\dot{x}(t) = f\big(x(t)\big)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ being continuously differentiable over $D = \{\|x\|_2 < r\}$ and $x(t) \in \mathbb{R}^n$, let its origin be an exponentially stable equilibrium point. Furthermore, let k, λ, and r_0 be positive constants subject to $r_0 < r/k$ such that $\|x(t)\|_2 \leq k\|x(0)\|_2 e^{-\lambda t}$ for all $x(0) \in D_0$ and $t \geq 0$, where $D_0 = \{\|x\|_2 < r_0\}$. Then, there is a continuously differentiable function $V(x)$ satisfying the inequalities given by*

$$c_1\|x\|_2^2 \leq V(x) \leq c_2\|x\|_2^2, \tag{1}$$

$$\frac{\partial V}{\partial x} f\big(x\big) \leq -c_3 \|x\|_2^2, \tag{2}$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 \leq c_4 \|x\|_2, \tag{3}$$

for all $x \in D_0$ with positive constants $c_1, c_2, c_3,$ and c_4. If, in addition, f is continuously differentiable for all x, globally Lipchitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in \mathbb{R}^n$.

3. Generalized Time Transformation Approach-Based Finite-Time Control

Consider the perturbed dynamical system given by

$$\dot{x}(t) = \alpha(t) f\big(x(t)\big) + g(t, x(t)), \quad x(0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\alpha(t) \in \mathbb{R}_+$ is a positive and time-varying scalar function entitled as "generalized finite-time gain function" (details below), $g(t, x(t)) \in \mathbb{R}^n$ is a bounded perturbation term satisfying $\|g(t, x(t))\|_2 \leq g^*$, and $f\big(x(t)\big)$ is a continuously differentiable and globally Lipschitz function. In addition, let the origin of the nominal dynamical system $\dot{x}(t) = f\big(x(t)\big)$ be globally exponentially stable. Note that the nominal dynamical system $\dot{x}(t) = f\big(x(t)\big)$ can represent a controlled dynamics and it can be also considered as the error dynamics resulting from an original baseline control algorithm, where the perturbation is set to zero and $\alpha(t)$ is neglected as $\alpha(t) = 1$. To elucidate the latter point, we now provide an example.

Example 1. *Consider a simple-yet-illustrative baseline scalar command following control algorithm given by $\dot{z}(t) = u(t)$ with $u(t) = -\big(z(t) - c(t)\big)$, where $z(t)$ is the state, $u(t)$ is the control, and $c(t)$ is a time-varying bounded command with bounded time rate of change. Defining the error as $x(t) \triangleq z(t) - c(t)$, one can write the corresponding error dynamics in the form given by $\dot{x}(t) = -x(t) - \dot{c}(t)$. If $c(t)$ is constant (i.e., $\dot{c}(t) = 0$), then the error dynamics* *reduces to* $\dot{x}(t) = -x(t)$*, where this is the so-called nominal dynamical system with* $f\big(x(t)\big) = -x(t)$ *for this example. Here, if we choose the control as* $u(t) = -\alpha(t)\big(z(t) - c(t)\big)$ *through multiplying the right hand side of the baseline algorithm with the generalized finite-time gain function* $\alpha(t)$*, we obtain its time-varying version as* $\dot{z}(t) = \alpha(t)\big(-z(t)+c(t)\big)$*. In this case, the resulting error dynamics can be written in the form given by (4); that is,* $\dot{x}(t) = \alpha(t)\big(-x(t)\big)-\dot{c}(t)$ *with* $g\big(t, x(t)\big) = -\dot{c}(t)$*.*

One of the objectives of this paper is to establish a class of generalized finite-time gain functions $\alpha(t)$ and the corresponding conditions in order to guarantee that the solution $x(t)$ of (4) converges to zero as $t \to \tau$, where $\tau \in \mathbb{R}_+$ is a user-defined convergence time. Motivated by this standpoint, we first introduce the following assumption.

Assumption 1. *The generalized finite-time gain function* $\alpha(t)$ *satisfies the following properties:*

- $\alpha(t)$ *is continuously differentiable on* $t \in [0, \tau)$*.*
- $\alpha(t) > m$ *for all* $t \in [0, \tau)$ *and for some* $m > 0$*.*
- $\lim_{t\to\tau} \alpha(t) = \infty$*.*

If one chooses a generalized finite-time gain function $\alpha(t)$ according to Assumption 1, then its corresponding generalized time transformation function $t = \theta(s)$ can be obtained based on the next lemma.

Lemma 2. *Consider a generalized finite-time gain function* $\alpha(t)$ *subject to Assumption 1 and the following conditions:*

i) $\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}(\theta(s))}{\mathrm{d}s} = \frac{1}{\alpha(\theta(s))}$ *(i.e,* $\alpha(\theta(s))\mathrm{d}(\theta(s)) = \mathrm{d}s$*).*

ii) $\theta(0) = 0$*.*

iii) $\lim_{s\to\infty} \theta(s) = \tau$*.*

If the generalized time transformation function $\theta(s)$ is obtained by solving the differential equation in i) along with the conditions ii) and iii), then the following statements hold:

a) $\theta(s)$ is continuously differentiable and strictly increasing over $s \in [0, \infty)$.

b) Let $h(s) \triangleq \frac{\mathrm{d}(\theta(s))}{\mathrm{d}s}$. Then, $h(s)$ is bounded and $\lim_{s\to\infty} h(s) = 0$. In addition, $\int_0^\infty h(r)\mathrm{d}r = \tau$.

Proof. To show *a)*, we need to show that $\mathrm{d}(\theta(s))/\mathrm{d}s$ is well-defined and greater than 0 over $s \in [0, \infty)$. We first note that the time transformation function $\theta(s)$ is actually a change of time from the stretched infinite-time interval $s \in [0, \infty)$ into the regular prescribed time interval $t \in [0, \tau)$; and vice versa. Therefore, $\alpha(t) = \alpha(\theta(s))$. In addition, from *i)*, we have

$$\frac{\mathrm{d}(\theta(s))}{\mathrm{d}s} = \frac{1}{\alpha(\theta(s))} = \frac{1}{\alpha(t)}. \tag{5}$$

Since $\alpha(t) > 0$ and it is continuously differentiable over $t \in [0, \tau)$, it is well-defined on $t \in [0, \tau)$; hence, $\mathrm{d}(\theta(s))/\mathrm{d}s > 0$ holds and also well-defined for all $s \in [0, \infty)$. As a result, $\theta(s)$ is continuously differentiable and strictly increasing over $s \in [0, \infty)$ (see, for example, [44]). Thus, the proof of *a)* is now complete.

To show *b)*, by the definition of $h(s)$ and (5), we have

$$h(s) = \frac{1}{\alpha(\theta(s))} = \frac{1}{\alpha(t)}. \tag{6}$$

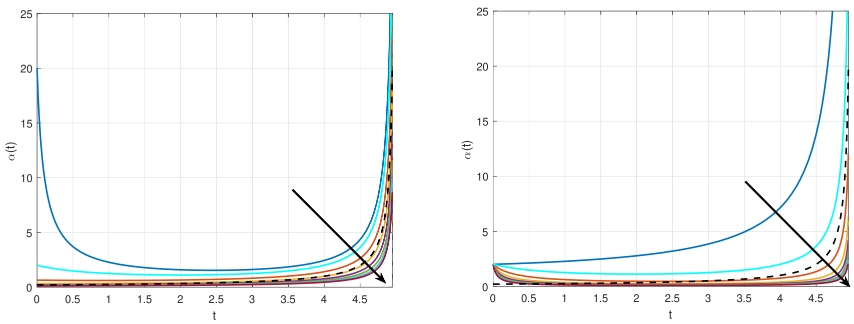
Since $\alpha(t)$ is positive definite and bounded from below by m, $h(s)$ is bounded from above by $h^* \triangleq m^{-1}$. In addition, *ii)* and *iii)* indicates that $t \to \tau$ as $s \to \infty$ and recall that $\alpha(t)$ satisfies $\lim_{t\to\tau} \alpha(t) = \infty$. Therefore, we have

$$\lim_{s\to\infty} h(s) = \lim_{s\to\infty} \frac{1}{\alpha\big(\theta(s)\big)} = \lim_{t\to\tau} \frac{1}{\alpha(t)} = 0. \tag{7}$$

Note also that

$$\int_0^\infty |h(r)|\mathrm{d}r = \int_0^\infty h(r)\mathrm{d}r = \int_0^\tau \mathrm{d}\big(\theta(r)\big) = \int_0^\tau \mathrm{d}t = \tau. \tag{8}$$

Thus, $h(s) \in \mathcal{L}_1$ on stretched infinite-time interval $s \in [0, \infty)$. Hence, the proof of *b)* is also now complete. ■



(a) $a \in [0.01, 2]$, $m = 0.1$, and $\tau = 5$.

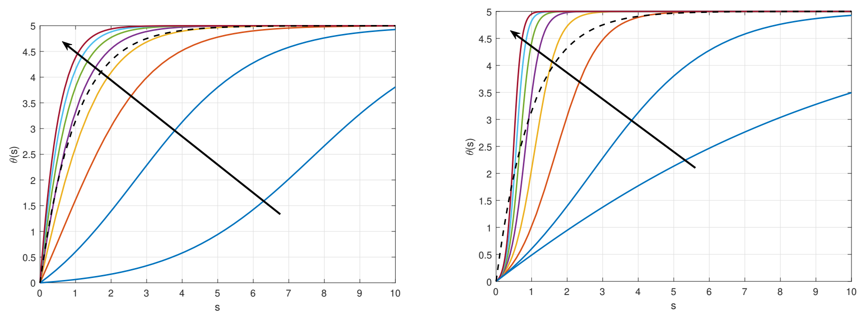
(b) $a = 0.1$, $m \in [0.01, 2]$ and $\tau = 5$.

Figure 1: Plots of the family of $\alpha(t)$ in Example 2 as a is increasing (left) and m is increasing (right), where the arrow pointing in the increasing direction of a and m. The dashed line represents $\alpha(t) = 1/(\tau - t)$ for comparison purpose.

Remark 1. *Lemma 2 establishes the theoretical connection between the generalized finite-time gain functions and their corresponding generalized time transformation functions. Therefore, in a reverse manner to Lemma 2, one can also start with a time transformation function $t = \theta(s)$ that satisfies the conditions ii) and iii) and the properties a) and b) of Lemma 2, and then solve the differential equation i) to obtain the generalized finite-time gain function $\alpha(t)$ subject to Assumption 1. As an example, the authors of [3] and [4] choose $\theta(s) = \tau(1 - e^{-s})$ and then obtain $\alpha(t) = 1/(\tau - t)$.*

In order to elucidate Lemma 2, we next provide candidate generalized finite-time gain functions $\alpha(t)$.

Example 2. *A common finite-time gain function is $\alpha(t) = 1/(\tau - t)$ with its corresponding time transformation function $\theta(s) = \tau(1 - e^{-s})$. We now consider a family of generalized finite-time gain functions defined by $\alpha(t) \triangleq \frac{1}{(\tau - t)(mt + a)}$, where $m \in \mathbb{R}_+$ and $a \in \mathbb{R}_+$. Here, $\alpha(t)$ satisfies the conditions in Assumption 1. After solving the differential equation i) of Lemma 2, one can obtain $\theta(s) = \frac{\tau a(e^{(a+m\tau)s} - 1)}{ae^{(a+m\tau)s} + m\tau}$ that satisfies the conditions ii) and iii). Figures 1 and 2 respectively show the plots of $\alpha(t)$ and the corresponding $\theta(s)$, as a is increasing while m is fixed (Figures 1(a) and 2(a)) and as m is increasing while*



(a) $a \in [0.01, 2]$, $m = 0.1$, and $\tau = 5$.

(b) $a = 0.1$, $m \in [0.01, 2]$, and $\tau = 5$

Figure 2: Plots of the family of $\theta(s)$ in Example 2 as a is increasing (left) and m is increasing (right), where the arrow pointing in the increasing direction of a and m. The dashed line represents $\theta(s) = \tau(1 - e^{-s})$ for comparison purpose.

a is fixed (Figures 1(b) and 2(b)). In these figures, the arrows point in the increasing direction of a and m. In addition, the common finite-time gain function $\alpha(t) = 1/(\tau - t)$ and its corresponding time transformation function $\theta(s) = \tau(1 - e^{-s})$ are also plotted for reference (dashed lines). Note that parameter a affects the initial gain of $\alpha(t)$ and parameter m affects the time rate of change of $\alpha(t)$ during the transient stage. Note also that different generalized finite-time gain functions $\alpha(t)$ lead to different transient behaviors of the system, we refer to Section 4 for an illustrative numerical example.

Building on the result of Lemma 2, we now show the convergence of the solution of the perturbed dynamical system given by (4) to zero over the prescribed regular time interval $[0, \tau)$.

Theorem 1. *Consider the perturbed dynamical system given by (4). If the generalized finite-time gain function $\alpha(t)$ satisfies Assumption 1 and there exists a corresponding generalized time transformation function $\theta(s)$ as stated in Lemma 2, then $\lim_{t \to \tau} x(t) = 0$.*

Proof. Since $x(t) = x(\theta(s))$, define $\bar{x}(s) \triangleq x(\theta(s))$. Then, the perturbed dynamical system given by (4) can be rewritten in the stretched infinite-time interval $s \in [0, \infty)$ as

$$\begin{aligned}\bar{x}'(s) \triangleq \frac{\mathrm{d}\bar{x}(s)}{\mathrm{d}s} =& \frac{\mathrm{d}\theta(s)}{\mathrm{d}s}\frac{\mathrm{d}\bar{x}(s)}{\mathrm{d}\theta(s)} \\ =& \frac{1}{\alpha(\theta(s))}\Big(\alpha(\theta(s))f\big(\bar{x}(s)\big) + g\big(\theta(s), \bar{x}(s)\big)\Big) \\ =& f\big(\bar{x}(s)\big) + \frac{1}{\alpha(\theta(s))}g\big(\theta(s), \bar{x}(s)\big) \\ =& f\big(\bar{x}(s)\big) + h(s)g\big(\theta(s), \bar{x}(s)\big),\ \bar{x}(0) = x_0, \end{aligned} \tag{9}$$

where $h(s) \triangleq 1/\alpha(\theta(s)) = 1/\alpha(t) = \mathrm{d}(\theta(s))/\mathrm{d}s$ as shown in Lemma 2. Define now $p(s, \bar{x}(s)) \triangleq h(s)g\big(\theta(s), \bar{x}(s)\big) \in \mathbb{R}^n$. Note that both $h(s)$ and $g\big(\theta(s), \bar{x}(s)\big)$ are respectively bounded by h^* and g^*; hence, $p(s, \bar{x}(s))$ is also bounded; that is, $\|p(s, \bar{x}(s))\|_2 \leq p^* \triangleq h^* g^*$. Now, one can rewrite (9) as

$$\bar{x}'(s) = f\big(\bar{x}(s)\big) + p(s, \bar{x}(s)), \quad \bar{x}(0) = x_0. \tag{10}$$

Since the origin of the nominal dynamical system $\dot{x}(t) = f\big(x(t)\big)$ of (4) is globally exponentially stable, the result of this theorem follows directly from Lemma 4.6 of [40]. Yet, we explicitly derive it here for the further analysis later. Specifically, for the nominal dynamical system, there exists a continuous function $V(x)$ satisfying the inequalities (1), (2) and (3) by Lemma 1. Utilizing this Lyapunov function and taking its derivative with respect to $s \in [0, \infty)$ along the trajectories of (10), we have

$$\begin{aligned} V'(\bar{x}) =& \frac{\partial V}{\partial \bar{x}}\Big(f\big(\bar{x}(s)\big) + p(s, \bar{x}(s))\Big) \\ =& \frac{\partial V}{\partial \bar{x}} f\big(\bar{x}(s)\big) + \frac{\partial V}{\partial \bar{x}} p(s, \bar{x}(s)) \\ \leq& -c_3\|\bar{x}(s)\|_2^2 + c_4\|\bar{x}(s)\|_2\|p(s, \bar{x}(s))\|_2 \\ \leq& -(1-\theta)c_3\|\bar{x}(s)\|_2^2 - \theta c_3\|\bar{x}(s)\|_2^2 + c_4\|\bar{x}(s)\|_2\|p(s, \bar{x}(s))\|_2 \\ \leq& -(1-\theta)c_3\|\bar{x}(s)\|_2^2,\ \forall\ \|\bar{x}(s)\|_2 \geq \frac{c_4\|p(s, \bar{x}(s))\|_2}{\theta c_3}, \end{aligned} \tag{11}$$

where $\theta \in (0, 1)$ and the third inequality comes from (2) and (3). By Theorem 4.19 of [40], the system (10) is input-to-state stable. Note that input-to-state stability implies that when the input converges to zero as $s \to \infty$, so does the state (see, for example, Exercise 4.58 in [40]). From Lemma 2, $\lim_{s\to\infty} h(s) = 0$;

hence, $\lim_{s\to\infty} p(s, \bar{x}) = 0$. As a result, $\bar{x}(s) \to 0$ as $s \to \infty$. Finally, since $t \to \tau$ as $s \to \infty$, $\lim_{t\to\tau} x(t) = 0$ follows. ∎

Remark 2. *Although the dynamical system given by (4) is perturbed, Theorem 1 shows that its state vector still converges to zero in a user-defined convergence time τ owing to the generalized finite-time gain function $\alpha(t)$. In addition, the perturbed dynamical system given by (4) often represents the error dynamics as discussed in Example 1 (see also, for example, [3, 4, 34] and references therein). In particular, the dynamics in, for example, [3, 4, 34] are linear; hence, the origins of their nominal systems are globally exponentially stable and readily satisfy the conditions of the perturbed dynamical system given by (4). Thus, for time-critical applications, if one designs a control algorithm for the dynamical system such that its error dynamics can be put into the form given by (4), its finite-time convergence is then guaranteed. To summarize, consistent with the discussion given in Example 1, the following three-step procedure can be adopted for designing a control algorithm for time-critical applications:*

- *Design a baseline control algorithm to exponentially satisfy the given objectives of a considered application over* $[0, \infty)$.
- *Find a generalized finite-time gain function $\alpha(t)$ that satisfies Assumption 1 and its corresponding generalized time transformation function $\theta(s)$ along the lines stated in Lemma 2.*
- *From the baseline control algorithm, obtain the error dynamics and compare it to (4). After that, come back to the baseline controller design process for introducing the generalized finite-time gain function $\alpha(t)$ and modify the controller such that the error dynamics has the form of (4).*

To illustrate the design procedure in Remark 2, we provide the following examples.

Example 3. *Consider the pendulum equation given by*

$$\ddot{\theta}(t) + \sin\theta(t) + b\dot{\theta}(t) = cu(t), \tag{12}$$

where b and c are positive coefficients. Suppose that the objective is to design a control signal $u(t)$ for the pendulum to reach an angle $\theta(t) = \delta$ at $t = \tau$. To address this problem, let $x_1(t) = \theta(t) - \delta$ and $x_2(t) = \dot{\theta}(t)$. Then, one can write

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), && (13) \\ \dot{x}_2(t) &= -\sin(x_1(t) + \delta) - b x_2(t) + cu, && (14) \end{aligned}$$

Following the design procedure proposed in Remark 2, we start with the baseline controller. Utilizing the backstepping design process with the auxiliary state $z(t) \triangleq k_x x_1(t) + x_2(t)$ and the Lyapunov function $V(x) = \frac{1}{2}x_1^2(t) + \frac{1}{2}z^2(t)$, one can readily obtain the intermediate control signal given by

$$u(t) = \frac{1}{c}\left(-x_1 - (k_x - b)(-k_x x_1(t) + z(t)) + \sin(x_1(t) + \delta) - k_z z(t)\right). \quad (15)$$

With this control signal, the system given by (13) and (14) becomes

$$\begin{aligned} \dot{x}_1(t) &= -k_x x_1(t) + z(t), && (16) \\ \dot{z}(t) &= -x_1 - k_z z(t). && (17) \end{aligned}$$

By choosing k_x and k_z such that the matrix $M \triangleq \begin{bmatrix} -k_x & 1 \\ -1 & -k_z \end{bmatrix}$ is Hurwitz, the origin of the system given by (16) and (17) becomes globally exponentially stable. We now put (16) and (17) into the form given by (4); that is,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{z}(t) \end{bmatrix} = \alpha(t) \begin{bmatrix} -k_x & 1 \\ -1 & -k_z \end{bmatrix} \begin{bmatrix} x_1(t) \\ z(t) \end{bmatrix}. \quad (18)$$

Next, we go back to the design process and redefine $z(t) \triangleq k_x x_1(t) + \frac{1}{\alpha(t)} x_2(t)$, and follow the same procedure to obtain the finite-time controller. The reason for revisiting the design process is that $\alpha(t)$ is a function of time; hence when introducing it into the auxiliary state $z(t)$ and taking the time derivative, the finite-time controller would contain the $\dot{\alpha}(t)$ term. Finally, the actual finite-time control signal that we obtain in this case has the form

$$u(t) = \frac{\alpha(t)}{c}\Big(- \alpha(t) x_1 - \Big(k_x - \frac{\dot{\alpha}(t)}{\alpha^2(t)} - \frac{b}{\alpha(t)}\Big)(-\alpha(t) k_x x_1(t) + \alpha(t) z(t))$$

$$+\frac{\sin(x_1(t)+\delta)}{\alpha(t)} - k_z\alpha(t)z(t)\Bigg). \tag{19}$$

Although the error dynamics given by (18) has no perturbations, we know from the result of Theorem 1 that the controller (19) can handle bounded perturbations.

To show the efficacy of the proposed approach, we consider the system given by (13) and (14) with $b = 1$, $c = 3$, $\delta = \frac{\pi}{2}$ *subject to the controller (19) with* $\alpha(t) = \frac{1}{(\tau-t)(mt+a)}$, *where* $m = 0, a = 0.25$ *and* $\tau = 1$, *and* $k_x = 1$, $k_z = 6$. *In addition, a constant perturbation term* $p = 2$ *is added into the right hand side of the dynamics of* $x_2(t)$ *given by (14). Figure 3 shows the closed-loop system responses with the proposed finite-time controller (19) for four different initial conditions. Specifically,* $x_1(t)$ *and* $z(t)$ *approach zero as* t *approaches* τ, *as expected. Thus, the objective that* $\theta(t)$ *approaches* δ *as* t *approaches* τ *is met. Yet, by definition* $z(t) = k_x x_1(t) + \frac{1}{\alpha(t)}x_2(t)$, $x_1(t)$ *and* $z(t)$ *approaching zero does not necessarily indicate that* $x_2(t)$ *also approaches zero. Conditions to guarantee the boundedness of the control algorithm are discussed in Theorem 2. Note that here we focus on developing control algorithms for applications that require the objectives to be met in a specific time interval, for*

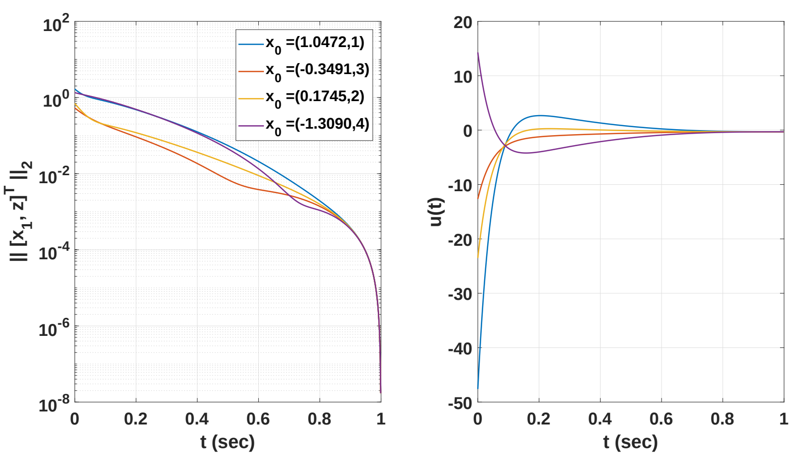


Figure 3: The norm $\|[x_1, z]^{\mathrm{T}}\|_2$ in semi-logarithmic scale for four different initial conditions (left) and the corresponding control inputs (right) of the pendulum system in Example 3.

instance, the missile engagement application, and this example shows that such finite-time control algorithms can be designed for a class of systems that can be stabilized via state feedback stabilization methods such as feedback linearization, backstepping, passivity-based control, and control Lyapunov functions, to name but a few examples. One future research direction can consider stabilizing the system beyond $t = \tau$*. To this end, we next briefly introduce a motivational example.*

Example 4. *In this example, we design a sliding mode controller that consists of two parts. The first part utilizes the result of Theorem 1 and the proposed design procedure in Remark 2 to develop a control algorithm for driving the dynamical system to the manifold* $z(t) = 0$ *in a user-defined finite time* τ *(reaching phase). The second part maintains the system on the manifold* $z(t) = 0$ *and stabilizes the origin of the system (sliding phase). For this purpose, consider the dynamical system given by*

$$\dot{x}_1(t) = x_1^2(t) - x_1^3(t) + x_2(t), \tag{20}$$

$$\dot{x}_2(t) = u(t). \tag{21}$$

Define now $z(t) \triangleq x_2(t) + x_1^2(t) + k_x x_1(t)$*, where* k_x *is a positive gain. Similar to Example 3, we utilize backstepping design method and the Lyapunov function* $V(x_1, z) = \frac{1}{2}x_1^2(t) + \frac{1}{2}z^2(t)$ *to obtain the finite-time controller for the reaching phase*

$$u(t) = -x_1(t) - (2x_1(t) + k_x)(-k_x x_1(t) - x_1^3(t) + z(t)) - k_z \alpha(t) z(t), \tag{22}$$

where k_z *is also a positive gain. With this controller, one can write*

$$\dot{z}(t) = -k_z \alpha(t) z(t) - x_1(t). \tag{23}$$

Since the controller (22) results in $\dot{V}(\cdot) < 0$ *on* $t \in [0, \tau)$*,* $x_1(t)$ *can be considered as a bounded perturbation in (23); hence, (23) has the form of (4). As a consequence, the result of Theorem 1 indicates that the controller (22) drives the system given by (20) and (21) to the manifold* $z(t) = x_2(t) + x_1^2(t) + k_x x_1(t) = 0$ *at* τ *time units.*

Next, suppose that the second state of the system subject to some bounded perturbation $p(t)$ such that $|p(t)| < \bar{p}$ and $\bar{p}$ is known. Here, one can use a standard method for designing the sliding mode controller as

$$u(t) = -\big(2x_1(t) + k_x\big)\big(-k_x x_1(t) - x_1^3(t) + z(t)\big) - k_z z(t) - \bar{p}\tanh(10z). \quad (24)$$

We note that the function $\operatorname{sign}(z)$ can be approximated by $\tanh(hx)$ with h being an appropriate positive number. Hence, we utilize $\tanh(10z)$ in (24) for reducing the chattering phenomenon. Combining (22) and (24), we obtain

$$u(t) = \begin{cases} -x_1(t) - (2x_1(t) + k_x)(-k_x x_1(t) - x_1^3(t) + z(t)) - k_z \alpha(t) z(t), & \text{if } t < \tau \\ -\big(2x_1(t) + k_x\big)\big(-k_x x_1(t) - x_1^3(t) + z(t)\big) - k_z z(t) - \bar{p}\tanh(10z). & \text{if } t \geq \tau \end{cases} \quad (25)$$

To show the effect of the controller (25), we consider $\alpha(t) = \frac{1}{(\tau - t)(mt+a)}$, where $m = 0, a = 0.25$ and $\tau = 0.25$, and $k_x = 2, k_z = 1$. In addition, a perturbation $p = 2\sin(t)$ is introduced into the dynamics of $x_2(t)$ given by (21). Figure 4 shows that the system reaches the manifold $z(t) = 0$ at $\tau = 0.25$ seconds as expected. For $t \geq \tau$, the states $x_1(t)$ and $x_2(t)$ are stabilized through the sliding mode controller. We note that although in this example there is only a subtle change in the controller $u(t)$ at time $t = \tau$ when the controller is switched, it is possible that there can be a jump. For such cases, a low-pass filter can be also utilized for the transition for avoiding the potential jump. In general, this example shows that the proposed finite-time controller can be used to accelerate (or decelerate) the stabilization process of the system over the infinite horizon when utilized in conjunction with an additional feedback controller. Once again, the stability analysis of the resulting closed-loop system under this kind of switching controller can be investigated as a future research and it goes beyond the results of this paper.

Remark 3. *From the third line in (11), we have*

$$V(\bar{x}(\infty)) - V(\bar{x}(0)) \leq -c_3 \int_0^\infty \|\bar{x}(r)\|_2^2 \mathrm{d}r + c_4 \int_0^\infty \|\bar{x}(r)\|_2 \|p\big(r, \bar{x}(r)\big)\|_2 \mathrm{d}r, \quad (26)$$

where $V(\bar{x}(\infty)) \triangleq \lim_{s\to\infty} V(\bar{x}(s)) = V(0) = 0$ from the result of Theorem 1. In addition, the result of Theorem 1 indicates that $\bar{x}(s)$ is bounded; hence, we

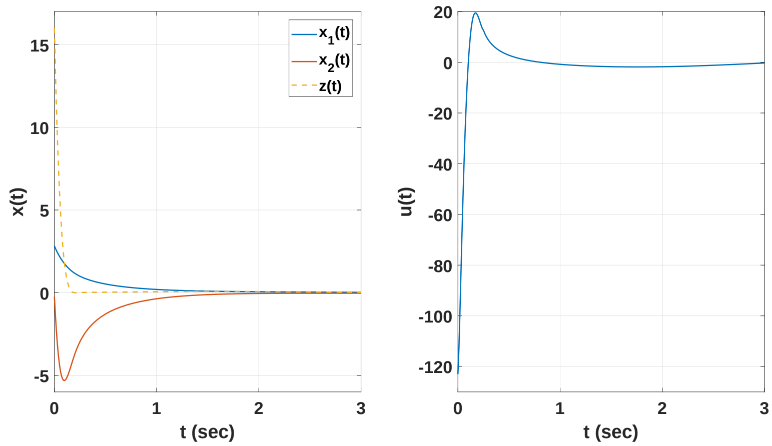


Figure 4: The state evolution of the system given by (20) and (21) (left) under the controller given by (25) (right) in Example 4.

can consider $\|\bar{x}(s)\|_2 \le \bar{x}^*$. *Therefore, from (26), one can write*

$$\begin{aligned}\int_0^\infty \|\bar{x}(r)\|_2^2 \mathrm{d}r &\le \frac{1}{c_3}\Big(V(\bar{x}(0)) + c_4\bar{x}^* \int_0^\infty \|p\big(r, \bar{x}(r)\big)\|_2 \mathrm{d}r\Big) \\ &\le \frac{1}{c_3}\Big(V(\bar{x}(0)) + c_4\bar{x}^* \int_0^\infty |h(r)|\|g\big(\theta(r), \bar{x}(r)\big)\|_2 \mathrm{d}r\Big) \\ &\le \frac{1}{c_3}\Big(V(\bar{x}(0)) + c_4\bar{x}^* g^* \int_0^\infty |h(r)|\mathrm{d}r\Big) \\ &= \frac{V(\bar{x}(0))}{c_3} + \frac{c_4\bar{x}^* g^* \tau}{c_3}, \end{aligned} \tag{27}$$

where the last equality results from (8). We note that the left hand side of (27) can be considered as an energy of $\bar{x}(s)$*, and therefore is of* $x(t)$*, and the right hand side of (27) is a constant. Thus,* $\bar{x}(s)$ *is a finite energy signal. The upper bound of this energy depends on* $V(\bar{x}(0))$ *and the product of* τ *with the bounds of both the state and the perturbation term. In particular, if the user-defined convergence time* τ *is selected to be large, then the upper bound of this energy given by the right hand side of (27) also increases to suppress the effect of the perturbation term. Yet,* $V(\bar{x}(0))$ *is independent of* τ *and can be interpreted as the energy required to drive the system from* $\bar{x}_0$ *to* 0*. In other words, when the perturbations on the dynamics are negligible, the system requires a fixed amount of energy to go from an initial value to zero equilibrium point regardless of the* *chosen τ. One can also consider the limit of a dynamical system's actuator when choosing τ. A workaround to prevent exceeding an actuator's performance is discussed in the next remark.*

Remark 4. *A practical approach to overcome the problem of exceeding actuator's performance is to first drive the system to a point or a region, where we know that it is feasible to achieve the objective in a user-defined convergence time τ without exceeding actuator's limit, and then activate the finite-time algorithm. Consider now that a feasible region Ψ is theoretically defined. The generalized finite-time gain function can then be redefined as*

$$\beta(t) \triangleq \begin{cases} \alpha(0), & x \notin \Psi, \\ \alpha(t - t_0), & x \in \Psi, \end{cases} \tag{28}$$

where t_0 is the time when the system enters the region Ψ providing that the system is capable of keeping track of execution time and detecting whether or not it is in Ψ. Note that $\beta(t)$ is a continuous function and identical to $\alpha(t)$ when $t_0 = 0$. By replacing $\alpha(t)$ by $\beta(t)$ in (4), the total execution time of the system is now $t_0 + \tau$. In particular, for the first t_0 time unit, the dynamics is time-invariant and is heading toward the feasible region Ψ. For $t \geq t_0$, the dynamics becomes time-varying and meets the objective in τ. An example of the feasible region is $\Psi = \{x(t) \in \mathbb{R}^n : \|x\|_\infty \leq x^*\}$ with $x^* \in \mathbb{R}_+$ being a known defined threshold.

Finally, the next theorem establishes the boundedness of $\dot{x}(t)$ over $t \in [0, \tau)$.[1]

Theorem 2. *Consider the perturbed dynamical system given by (4). Consider, in addition, the following conditions:*

i) $\frac{\dot{\alpha}(t)}{\alpha^2(t)}$ *is bounded on* $t \in [0, \tau)$, *and* $\lim_{t \to \tau} \frac{\dot{\alpha}(t)}{\alpha^2(t)} = \kappa < \infty$.

[1]Similar to the discussion in Example 1, if we consider the control signal as $u(t) = \alpha(t) f\big(x(t)\big)$, then it is also bounded as a result of the following theorem as well as from the boundedness of $g\big(t, x(t)\big)$.

ii) $\bar{r}'(s) = \left(\frac{\mathrm{d}f\left(\bar{x}(s)\right)}{\mathrm{d}\bar{x}} + \frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)} h^2(s)\mathrm{I}_n\right)\bar{r}(s)$ *is globally exponentially stable, where* $r(t) = r\big(\theta(s)\big)$ *and* $\bar{r}(s) \triangleq r\big(\theta(s)\big)$.

Then, $\dot{x}(t)$ *is bounded for* $t \in [0, \tau)$.

Proof. From (4), if we show that $r(t) \triangleq \alpha(t) f\big(x(t)\big)$ is bounded over $t \in [0, \tau)$, then we can directly conclude that $\dot{x}(t)$ is bounded over $t \in [0, \tau)$ since $g\big(t, x(t)\big)$ is bounded over $t \in [0, \tau)$. For this purpose, we now write the time derivative of $r(t)$ as

$$\begin{aligned}
\dot{r}(t) &= \dot{\alpha}(t) f\big(x(t)\big) + \alpha(t)\dot{f}\big(x(t)\big) \\
&= \dot{\alpha}(t) f\big(x(t)\big) + \alpha(t)\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \dot{\alpha}(t)\left(\frac{1}{\alpha(t)} r(t)\right) + \alpha(t)\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x}\big(r(t) + g(t, x(t))\big) \\
&= \left(\alpha(t)\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x} + \frac{\dot{\alpha}(t)}{\alpha(t)}\mathrm{I}_n\right) r(t) + \alpha(t)\frac{\mathrm{d}f\big(x(t)\big)}{\mathrm{d}x} g(t, x(t)),
\end{aligned} \tag{29}$$

where the third equality comes from the definition of $r(t)$ and (4). Since $r(t) = r\big(\theta(s)\big)$, define $\bar{r}(s) \triangleq r\big(\theta(s)\big)$. Similar to the proof of Theorem 1, one can rewrite (29) in the stretched infinite-time interval $s \in [0, \infty)$ as

$$\bar{r}'(s) = \left(\frac{\mathrm{d}f\big(\bar{x}(s)\big)}{\mathrm{d}\bar{x}} + \frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)} h^2(s)\mathrm{I}_n\right)\bar{r}(s) + \frac{\mathrm{d}f\big(\bar{x}(s)\big)}{\mathrm{d}\bar{x}} g\big(\theta(s), \bar{x}(s)\big), \tag{30}$$

where $h(s) \triangleq 1/\alpha(\theta(s))$. Since $f\big(x(t)\big)$ is globally Lipschitz, the second term of (30) is bounded. In addition, by conditions *i)* and *ii)*, we conclude that $\bar{r}(s)$ is a bounded solution to the dynamical system (30) on the stretched infinite-time interval $s \in [0, \infty)$ (see Remark 5 for more explanation on conditions *i)* and *ii)*). Therefore, $r(t)$ is bounded over $t \in [0, \tau)$, where this implies the boundedness of $\dot{x}(t)$ over $t \in [0, \tau)$. ■

Remark 5. *In the proof of Theorem 2,* $\dot{x}(t)$ *is bounded when* $r(t) = \alpha(t) f\big(x(t)\big)$ *is bounded. Note that* $\lim_{t \to \tau} \alpha(t) = \infty$ *by Assumption 1 and* $\lim_{t \to \tau} f\big(x(t)\big) = 0$ *by Theorem 1. Therefore,* $r(t)$ *is bounded when* $f\big(x(t)\big)$ *converge to zero before* $\alpha(t)$ *becomes very large. Indeed, conditions i) and ii) of Theorem 2 capture this* *phenomenon. Specifically, the term* $\alpha(t)\frac{\mathrm{d}f\left(x(t)\right)}{\mathrm{d}x}$ *in (29) represents the rate of change of* $r(t)$ *along* $f\left(x(t)\right)$ *and the term* $\frac{\dot{\alpha}(t)}{\alpha(t)}\mathrm{I}_n$ *in (29) is the rate of change of* $r(t)$ *along* $\alpha(t)$*. When transforming these terms into the stretched infinite-time interval* $s \in [0, \infty)$*, the condition ii) apprehends the above requirement. In addition, we note that*

$$\frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)} = \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = \dot{\alpha}(t), \tag{31}$$

while $h^2(s) = 1/\alpha^2(t)$*; hence,* $\frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)}h^2(s) = \frac{\dot{\alpha}(t)}{\alpha^2(t)}$*. This induces condition i). We refer to Remark 7 and Figure 5 for an illustration of this point.*

Remark 6. *The conditions i) and ii) of Theorem 2 are the generalized forms of the conditions in, for example, [3, 4] and [45]. Specifically, in these papers,* $\alpha(t) = 1/(\tau - t)$ *and* $\dot{\alpha}(t) = \alpha^2(t)$*; hence, the condition i) automatically holds. In addition, these papers consider linear systems; thus,* $\mathrm{d}f\left(\bar{x}(s)\right)/\mathrm{d}\bar{x}$ *is often depicted by a Hurwitz matrix such as* $\mathrm{d}f\left(\bar{x}(s)\right)/\mathrm{d}\bar{x} = \gamma M$ *with* $\gamma \in \mathbb{R}_+$ *being a design parameter. As a consequence, assumption ii) simplifies to the requirement that the matrix* $(\gamma M + \mathrm{I}_n)$ *is Hurwitz. In this case, since* M *is Hurwitz, it is straightforward to show that by choosing* $\gamma > -1/\operatorname{Re}\left(\lambda_{\max}(M)\right)$*, the matrix* $(\gamma M + \mathrm{I}_n)$ *is guaranteed to be Hurwitz.*

Remark 7. *Note for the candidate family of generalized finite-time gain functions utilized in Example 2,* $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$*, that* $\dot{\alpha}(t) = (2mt - m\tau + a)\alpha^2(t)$*; hence, the condition i) of Theorem 2 is satisfied with* $\kappa = m\tau + a$ *and this is also the upper bound of* $\dot{\alpha}(t)/\alpha^2(t)$ *over* $t \in [0, \tau)$*. Note also that* $\lim_{s\to\infty} \frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)}h^2(s) = \kappa$*. To illustrate this point, Figure 5 shows the plot of* $\frac{\dot{\alpha}(t)}{\alpha^2(t)}$ *over the regular prescribed time interval* $[0, \tau]$ *(left) and the plot of its identical version* $\frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)}h^2(s)$ *over the stretched infinite-time interval* $[0, \infty)$ *(right) with* $a = 0.1$*,* $m = 0.1$*, and* $\tau = 5$*, and the dashed line represents* κ*. In addition, similar to Remark 6, as applied to linear systems with* $\mathrm{d}f\left(\bar{x}(s)\right)/\mathrm{d}\bar{x} \triangleq \gamma M$ *being a Hurwitz matrix, the assumption ii) is satisfied when the matrix* $(\gamma M + \kappa\mathrm{I}_n)$ *is Hurwitz (Interested readers can refer to Example 9.6,*

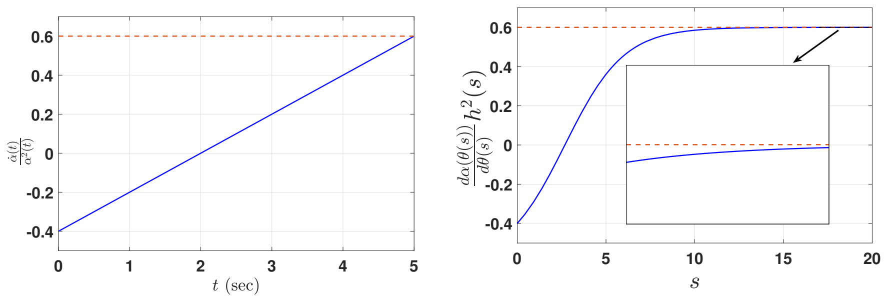


Figure 5: Plot of $\frac{\dot{\alpha}(t)}{\alpha^2(t)}$ over the regular prescribed time interval $[0, \tau]$ (left) and plot of its identical version $\frac{\mathrm{d}\alpha(\theta(s))}{\mathrm{d}\theta(s)}h^2(s)$ over the stretched infinite-time interval $[0, \infty)$ (right) for $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$ with $a = 0.1$, $m = 0.1$, and $\tau = 5$. The dashed line represents the upper bound κ.

Corollary 9.1 and Lemma 9.5 of [40] for similar analysis). Once again, since M is Hurwitz, when $\gamma > -\kappa / \operatorname{Re}\big(\lambda_{\max}(M)\big)$, $(\gamma M + \kappa \mathrm{I}_n)$ is guaranteed to be Hurwitz. Note that in Example 3, $\gamma = 1$ and the matrix $M \triangleq \begin{bmatrix} -k_x & 1 \\ -1 & -k_z \end{bmatrix}$ with $k_x = 1$ and $k_z = 6$ has $\lambda_{\max}(M) = -1.2087$. In addition, with the choice of $\alpha(t) = \frac{1}{(\tau-t)(mt+a)}$, where $m = 0, a = 0.25$ and $\tau = 1$, we have $\kappa = 0.25$ and the condition $\gamma > -\kappa / \operatorname{Re}\big(\lambda_{\max}(M)\big) = 0.2068$ is readily satisfied. Hence, $\begin{bmatrix} \dot{x}_1^{\mathrm{T}}(t) & \dot{z}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$ is bounded, which indicates that the controller $u(t)$ given by (19) is also uniformly bounded on $t \in [0, \tau)$.

4. Finite-Time Distributed Control of Networked Multiagent Systems

During the last two decades, networked multiagent systems have started to become a relatively mature research field addressing important problems through local interactions such as consensus, leader-follower, flocking, formation control, containment control; to name but a few examples (see, for example, [42, 46, 47, 48] and references therein). In this section, we present a theoretical application of our findings in Section 3 to the distributed control of networked multiagent systems to illustrate their effectiveness. In particular, we consider the leader-follower problem with a networked multiagent system containing N agents subject to a connected and undirected graph $\mathcal{G}$. To this end, let the baseline distributed control algorithm of agents be (see, for example, [49])

$$\dot{x}_i(t) \;=\; -\gamma\left(\sum_{i\sim j}\big(x_i(t)-x_j(t)\big)+k_i\big(x_i(t)-c(t)\big)\right), \quad x_i(0)=x_{i0}, \tag{32}$$

where $x_i(t)\in\mathbb{R}$ is the state of agent i, $i=1,\ldots,N$, $c(t)\in\mathbb{R}$ is the bounded command with bounded time rate of change, $\gamma\in\mathbb{R}_+$ is a scalar gain, $k_i=1$ if agent i is the leader, and $k_i=0$ otherwise.

Defining the error as $\tilde{x}_i(t)\triangleq x_i(t)-c(t)$, one can write

$$\dot{\tilde{x}}_i(t) \;=\; -\gamma\left(\sum_{i\sim j}\big(\tilde{x}_i(t)-\tilde{x}_j(t)\big)+k_i\tilde{x}_i(t)\right)-\dot{c}(t), \quad \tilde{x}_i(0)=\tilde{x}_{i0}. \tag{33}$$

By defining $\tilde{x}(t)\triangleq[\tilde{x}_1(t),\ldots,\tilde{x}_N(t)]^{\mathrm{T}}$, one can obtain the compact form of the error dynamics in (33) as

$$\dot{\tilde{x}}(t)=-\gamma\left(\mathcal{L}(\mathcal{G})+K\right)\tilde{x}(t)-\mathbf{1}_N\dot{c}(t), \quad \tilde{x}(0)=\tilde{x}_0, \tag{34}$$

where $\mathcal{L}(\mathcal{G})\in\mathbb{R}^{N\times N}$ is the Laplacian matrix of the communication graph $\mathcal{G}$ and $K=\mathrm{diag}([k_1,k_2,\ldots,k_N]^{\mathrm{T}})$. Here, $-\left(\mathcal{L}(\mathcal{G})+K\right)$ is a Hurwitz matrix (see, for example, Lemma 3.3 of [49]).

Next, by multiplying the baseline algorithm in (32) with the generalized finite-time gain function $\alpha(t)$, we obtain a time-varying distributed control algorithm for time-critical applications in the form

$$\dot{x}_i(t)=u_i(t), \quad x_i(0)=x_{i0}, \tag{35}$$

$$u_i(t)=-\gamma\alpha(t)\left(\sum_{i\sim j}\big(x_i(t)-x_j(t)\big)+k_i\big(x_i(t)-c(t)\big)\right). \tag{36}$$

In this case, the resulting error dynamics becomes

$$\dot{\tilde{x}}(t)=-\gamma\alpha(t)\left(\mathcal{L}(\mathcal{G})+K\right)\tilde{x}(t)-\mathbf{1}_N\dot{c}(t), \quad \tilde{x}(0)=\tilde{x}_0, \tag{37}$$

which clearly satisfies the form of the perturbed dynamical system given by (4) with $f\big(x(t)\big)=-\gamma\left(\mathcal{L}(\mathcal{G})+K\right)\tilde{x}(t)$ and $g\big(t,x(t)\big)=-\mathbf{1}_N\dot{c}(t)$. The next corollary is now immediate.

Corollary 1. *Consider the networked multiagent system with N agents given by (35) and (36), where agents exchange information using their local measurements through an undirected and connected graph topology $\mathcal{G}$. In addition, consider that the generalized finite-time gain function $\alpha(t)$ satisfies Assumption 1, $\lim_{t\to\tau} \frac{\dot{\alpha}(t)}{\alpha^2(t)} = \kappa < \infty$, and there exists a corresponding generalized time transformation function $\theta(s)$ as stated in Lemma 2. By defining $M \triangleq -(\mathcal{L}(\mathcal{G}) + K)$ and choosing $\gamma > -\kappa/\operatorname{Re}(\lambda_{\max}(M))$, then $\lim_{t\to\tau} x(t) = c(\tau)$ holds and $\dot{x}(t)$ is bounded for $t \in [0, \tau)$.*

Proof. The results directly follows from Theorems 1 and 2 as well as Remark 7. ■

We are now ready to present an example to illustrate the result in Corollary 1.

Example 5. *In this example, we consider a multiagent system under the algorithm given by (35) and (36) with 5 agents subject to a ring graph, where the first agent is the leader and the rest are followers. Here, we choose the user-defined convergence time as $\tau = 5$ seconds and the command as $c(t) = 5 + 0.75\sin(t)$. Along the lines of the discussion in Example 2 of Section 3, the family of generalized finite-time gain functions defined by $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$ satisfies Assumption 1 and there exists a corresponding family of generalized time transformation functions $\theta(s)$ as stated in Lemma 2. In what follows, we consider the two cases: We choose $a = 0.5$ and $m = 0.005$ for the first case and choose $a = 0.1$ and $m = 0.085$ for the second case. Based on Remark 7, note that $M = -(\mathcal{L}(\mathcal{G}) + K)$ is Hurwitz and the upper bounded of $\dot{\alpha}(t)/\alpha^2(t)$ on $t \in [0, \tau)$ is $\kappa = m\tau + a = 0.525$ for both cases; hence, we obtain $-\kappa/\lambda_{\max}(M) = 3.7717$. By choosing $\gamma = 4$ for both cases, the assumptions of Theorem 2 are now satisfied; hence, $\dot{x}(t)$ is bounded over $t \in [0, \tau)$. The same random initial conditions for agents are utilized for both cases.*

Figure 6 shows the response of the networked multiagent system under the control algorithm given by (36) with $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$, $a = 0.5$, $m = 0.005$, $\tau = 5$ seconds, and $\gamma = 4$, where the solid lines are the states of agents (left) and their time derivative (right), and the dashed line shows the command. Similarly,

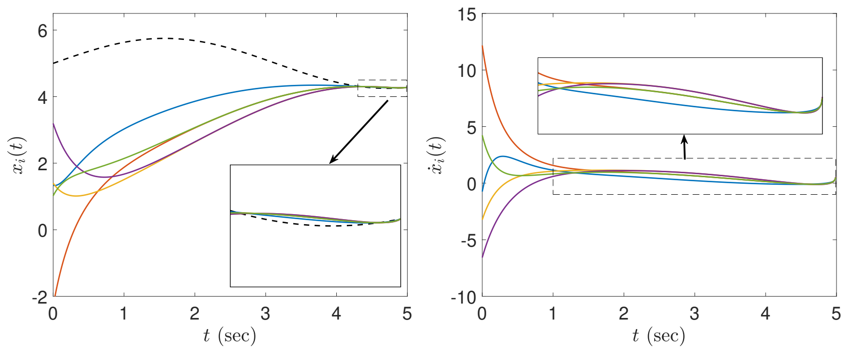


Figure 6: Response of the networked multiagent system under algorithm (36) with $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$, $a = 0.5$, $m = 0.005$, $\tau = 5$ seconds, and $\gamma = 4$, where the solid lines are the states of agents (left) and the time derivative of agents (right), and the dashed line shows the tracking command.

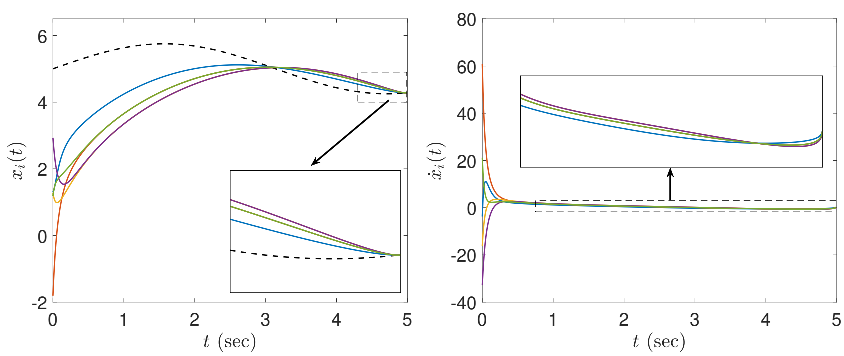


Figure 7: Response of the networked multiagent system under algorithm (36) with $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$, $a = 0.1$, $m = 0.085$, $\tau = 5$ seconds, and $\gamma = 4$, where the solid lines are the states of agents (left) and the time derivative of agents (right), and the dashed line shows the tracking command.

Figure 7 shows the response of the networked multiagent system under algorithm (36) with $\alpha(t) \triangleq \frac{1}{(\tau-t)(mt+a)}$, $a = 0.1$, $m = 0.085$, $\tau = 5$ *seconds, and* $\gamma = 4$, *where the solid lines are the states of agents (left) and their time derivative (right), and the dashed line shows the command. As expected from the result of Corollary 1, the states of all agents approach to the command* $c(t)$ *as* $t \to \tau = 5$ *seconds. Note that when* $t = 0$, $\alpha(0) = 1/(\tau a)$; *hence, the parameter*

a affects the initial value of $\alpha(t)$. Moreover, from discussion in Remark 7, $\dot{\alpha}(t) = (2mt - m\tau + a)\alpha^2(t)$; hence, m affects the time rate of change of $\alpha(t)$. Since the second case has a smaller value for a, the initial value of $\alpha(t)$ in the second case is larger than in the first case. This is depicted by the higher initial values of $\dot{x}(t)$ in the right plot of Figure 7 compared to the one in Figure 6. Finally, as expected from Theorem 2, the right plots show that $\dot{x}_i(t)$ remains bounded over $t \in [0, \tau)$. In general, different values of a and m in both cases lead to different transient behaviors of the resulting networked multiagent system.

5. Conclusion

For contributing to the recent studies on finite-time control based on the time transformation approach, we investigated a new class of scalar, time-varying gain functions entitled as "generalized finite-time gain functions." We showed how these functions have the capability to convert an original baseline control algorithm into a time-varying one to allow the system to be executed over a prescribed time interval. We also rigorously showed all the stability conditions with regard to the proposed family of generalized finite-time gain functions that guarantee the boundedness and convergence of the state and control signals when the considered class of dynamical systems are subject to perturbations. The results were then utilized to propose a procedure for designing finite-time algorithms and illustrated with numerical examples. This indicates that the proposed method is applicable to, but not limited to, a class of nonlinear systems and multiagent systems. Finally, we presented an application of our theoretical findings to distributed control of networked multiagent systems over a prescribed time interval.

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